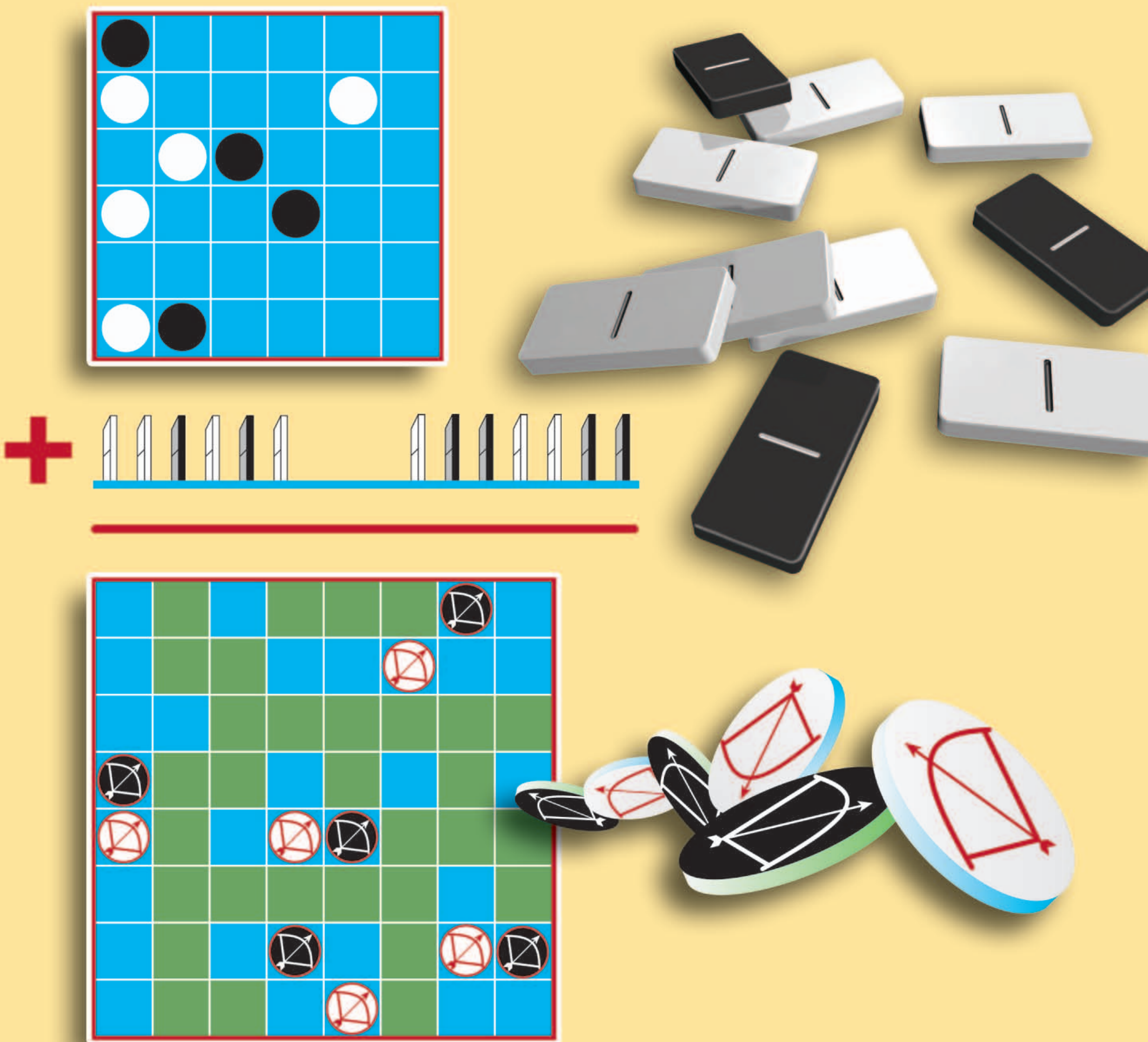
LESSONS IN PLAY

An Introduction to Combinatorial Game Theory



MICHAEL H. ALBERT • RICHARD J. NOWAKOWSKI • DAVID WOLFE

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To Richard K. Guy, a gentleman and a mathematician

Contents

Preface

It should be noted that children's games are not merely games. One should regard them as their most serious activities.

Michel Eyquem de Montaigne

Herein we study games of pure strategy, in which there are only two players[1] who alternate moves, without using dice, cards or other random devices and where the players have perfect information about the current state of the game. Familiar games of this type include: TIC TAC TOE, DOTS & BOXES, CHECKERS and CHESS. Obviously, card games such as GIN RUMMY, and dice games such as BACKGAMMON are not of this type. The game of BATTLESHIP has alternate play, and no chance elements, but fails to include perfect information — in fact that's rather the point of BATTLESHIP. The games we study have been dubbed *combinatorial games* to distinguish them from the games usually found under the heading of *game theory,* which are games that arise in economics and biology.

For most of history, the mathematical study of games consisted largely of separate analyses of extremely simple games. This was true up until the 1930s when the Sprague-Grundy theory provided the beginnings of a mathematical foundation for a more general study of games. In the 1970s, the twin tomes *On Numbers and Games* by Conway and *Winning Ways* by Berlekamp, Conway, and Guy established and publicized a complete and deep theory, which can be deployed to analyze countless games. One cornerstone of the theory is the notion of a disjunctive sum of games, introduced by John Conway for normal-play games. This scheme is particularly useful for games that split naturally into components. *On Numbers and Games* describes these mathematical ideas at a sophisticated level. *Winning Ways* develops these ideas, and many more, through playing games with the aid of many a pun and witticism. Both books

[1]In 1972, Conway's first words to one of the authors, who was an undergraduate at the time, was "What's $1+1+1$?" alluding to three-player games. This question has still not been satisfactorily answered.

have a tremendous number of ideas and we acknowledge our debt to the books and to the authors for their kind words and teachings throughout our careers.

The aim of our book is less grand in scale than either of the two tomes. We aim to provide a guide to the evaluation scheme for normal-play, two-player, finite games. The guide has two threads, the theory and the applications.

The theory is accessible to any student who has a smattering of general algebra and discrete math. Generally, a third year college student, but any good high school student should be able to follow the development with a little help. We have attempted to be as complete as possible, though some proofs in Chapters 8 and 9 have been omitted, because the theory is more complex or is still in the process of being developed. Indeed, in the last few months of writing, Conway prevailed on us to change some notation for a class of all-small games. This *uptimal* notation turned out to be very useful and it makes its debut in this book.

We have liberally laced the theory with examples of actual games, exercises and problems. One way to understand a game is to have someone explain it to you; a better way is to muse while pushing some pieces around; and the best way is to play it against an opponent. Completely solving a game is generally hard, so we often present solutions to only some of the positions that occur within a game. The authors invented more games than they solved during the writing of this book. While many found their way into the book, most of these games never made it to the rulesets found at the end. A challenge for you, the reader of our missive, and as a test of your understanding, is to create and solve your own games as you progress through the chapters.

Since the first appearance of *On Numbers and Games* and *Winning Ways* there have been several conferences specifically on combinatorial games. The subject has moved forward and we present some of these developments. However, the interested reader will need to read further afield to find the theories of loopy games, misère-play games, other (non-disjunctive) sums of games, and the computer science approach to games. The proceedings of these conferences [Guy91, Now96, Now02, FN04] would be good places to start.

Organization of the Book

The main idea of this part of the theory of combinatorial games is the assigning of values to games, values that can be used to replace the actual games when deciding who wins and what the winning strategies might be.

Each chapter has a prelude which includes problems for the student to use as a warm-up for the mathematics to be found in the following chapter. The prelude also contains guidance to the instructor for how one can wisely deviate from the material covered in the chapter.

Exercises are sprinkled throughout each chapter. These are intended to reinforce, and check the understanding of, the preceding material. Ideally then, a student should try every exercise as it is encountered. However, there should be no shame associated with consulting the solutions to the exercises found at the back of the book if one or more of them should prove to be intractable. If that still fails to clear matters up satisfactorily, then it may be time to consult a *games guru.*

Chapter 0 introduces basic definitions and loosely defines that portion of game theory which we will address in the book. Chapter 1 covers some general strategies for playing or analyzing games and is recommended for those who have not played many games. Others can safely skim the chapter and review sections on an as-needed basis while reading the body of the work. Chapters 2, 4, and 5 contain the core of the general mathematical theory. Chapter 2 introduces the first main goal of the theory, that being to determine a game's *outcome class* or who should win from any position. Curiously, a great deal of the structure of some games can be understood solely by looking at outcome classes. Chapter 3 motivates the direction the theory takes next. Chapters 4, 5, and 6 then develop this theory (i.e., assigning values and the consequences of these values.)

Chapters 7, 8, and 9 look at specific parts of the universe of combinatorial games and as a result, these are a little more challenging but also more concrete since they are tied more closely to actual games. Chapter 7 takes an in-depth look at *impartial* games. The study of these games pre-dates the full theory. We place them in the new context and show some of the new classes of games under present study. Chapter 8 addresses hot games, games such as GO and AMAZONS in which there is a great incentive to move first. There is much research in this area and we can only give an introduction to this material. Chapter 9 looks at the analysis of *all-small* games. Most of the research emphasis has been on impartial and hot games. Only recently have there been developments in this area and we present the original and latest results in light of all the new developments in combinatorial game theory.

Chapter ω is a brief listing of other areas of active research that we could not fit into an introductory text.

In Appendix A, we present top-down induction, an approach that we use often in the text. While the student need not read the appendix in its entirety, the first few sections will help ground the format and foundation of the inductive proofs found in the text.

Appendix B is a brief introduction to CGSuite, a powerful programming toolkit written by Aaron Siegel in Java for performing algebraic manipulations on games. CGSuite is to the combinatorial game theorist what Maple or Mathematica is to a mathematician or physicist. While the reader need not use CGSuite while working through the text, the program does help to build intuition, double-check work done by hand, develop hypotheses, and handle some of the drudgery of rote calculations.

Appendix D contains the rules to any game in the text that either appears multiple times or is found in the literature. We do not always state the rules to a game within the text, so the reader will want to refer to this appendix often.

The supporting website for the book is located at www.lessonsinplay.com. Look there for links, programs, and addenda, as well as instructions for accessing the online solutions manual for instructors.

Acknowledgments

While we are listed as the *authors* of this text, we do not claim to be the main contributors. The textbook emerged from a mathematically rich environment created by others. We got to choose the words and consequently, despite the best efforts of friends and colleagues, all the errors are ours.

Many of the contributors to this environment are cited within the book. There were many that also contributed to and improved the contents of the text itself and who deserve special thanks. We are especially grateful to Elwyn Berlekamp, John Conway, and Richard Guy who encouraged — and, at times, hounded — us to complete the text, and we hope it helps spawn a new generation of active aficionados.

Naturally, much of the core material and development is a reframing of material in *Winning Ways* and *On Number and Games.* We have adopted some of the proofs of J. P. Grossman, particularly that of the *Number-Avoidance Theorem.* Aviezri Fraenkel contributed the *Fundamental Theorem of Combinatorial Games,* which makes its appearance at the start of Chapter 2. Dean Hickerson helped us to prove Theorem 6.19 on page 126, that a game with negative incentives must be a number. Conway repeatedly encouraged us to adopt the *uptimal* notation in Chapter 9, and it took us some time to see the wisdom of his suggestions. Elwyn Berlekamp and David Molnar contributed some fine problems. Paul Ottaway, Angela Siegel, Meghan Allen, Fraser Stewart, and Neil McKay were students who pretested portions of the book and provided useful feedback, corrections, and clarifications. Elwyn Berlekamp, Richard Guy, Aviezri Fraenkel, and Aaron Siegel edited various chapters of our work for technical content, while Christine Aikenhead edited for language. Brett Stevens and Chris Lewis read and commented on parts of the book. Susan Hirshberg contributed the title of our book.

In this age of large international publishers, A K Peters is a fantastic and refreshing publishing house to work with. While they appreciate and under-

stand the business of publishing, we are convinced they care more about the dissemination of fine works than about the bottom line.

We thank our spice[2] for their loving support, and Lila and Tovia, who are the real *Lessons in Play.*

[2]affectionate plural of spouse

Preparation for Chapter 0

Before each chapter are several quick prep problems which are worth tackling in preparation to reading the chapter.

Prep Problem 0.1. List all the two-player games you know which do not involve chance (dice or coin flips).

Prep Problem 0.2. Locate the textbook website, www.lessonsinplay.com, and determine whether it might be of use to you.

To the instructor: Before each chapter, we will include a few suggestions to the instructor. Usually these will be examples which do not appear in the book, but which may be worth covering in lecture. The student unsatisfied by the text may be equally interested in seeking out these examples.

We highly recommend that the instructor and the student read Appendix A on top-down induction. We present induction in a way that will be unfamiliar to most, but which leads to more natural proofs, particularly those found in combinatorial game theory.

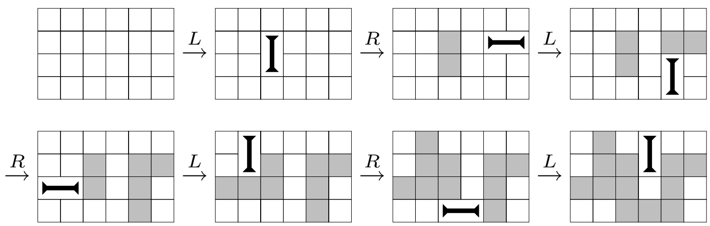
The textbook website, www.lessonsinplay.com, has instructions for how instructors can obtain a solution manual.

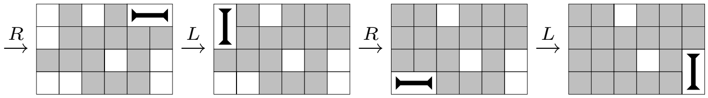
Chapter 0
Combinatorial Games

We don't stop playing because we grow old;
we grow old because we stop playing.

George Bernard Shaw

This book is all about *combinatorial games* and the mathematical techniques that can be used to analyze them. One of the reasons for thinking about games is so that you can be more skillful and have more fun playing them; so let's begin with an example called DOMINEERING. To play you will need a chessboard and a set of dominoes. The domino pieces should be big enough to cover or partially cover two squares of the chessboard but no more. You can make do with a chessboard and some slips of paper of the right size or even play with pen or pencil on graph paper (but the problem there is that it will be hard to undo moves when you make a mistake!). The rules of DOMINEERING are simple. Two players alternately place dominoes on the chessboard. A domino can only be placed so that it covers two adjacent squares. One player, Louise, places her dominoes so that they cover vertically adjacent squares. The other player, Richard, places his dominoes so that they cover horizontally adjacent squares. The game ends when one of the players is unable to place a domino, and that player then loses. Here is a sample game on a 4×6 board with Louise moving first:

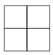
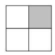




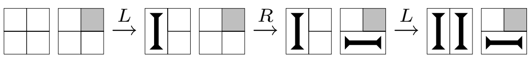
Since Louise placed the last domino, she has won.

Exercise 0.1. Stop reading! Find a friend and play some games of DOMINEERING. A game on a full chessboard can last a while so you might want to play on a 6×6 square to start with.

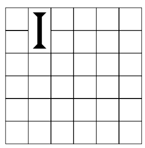
If you did the exercise then you probably made some observations and learned a few tactical tricks in DOMINEERING. One observation is that after a number of dominoes have been placed the board *falls apart* into disconnected regions of empty squares. When you make a move you need to decide what region to play in and how. Suppose that you are the vertical player and that there are two regions of the form:

 and 

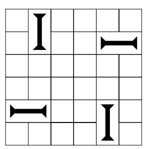
Obviously you could move in either region. However, if you move in the hook-shaped region then your opponent will move in the square. You will have no more moves left so you will lose. If instead you move in the square, then your opponent's only remaining move is in the hook. Now you still have a move in the square to make, and so your opponent will lose. If you are L and your opponent is R play should proceed



This is also why an opening move such as



is good since it *reserves* the two squares in the upper left for you later. In fact, if you play seriously for a while it is quite possible that the board after the first four moves will look something like:



Simply put, the aim of combinatorial game theory is to understand in a more detailed way the principles underlying the sort of observations we have just made about DOMINEERING. We will learn about games in general and how to understand them but, as a bonus, how to play them well!

0.1 Basic Terminology

In this section we will provide an informal introduction to some of the basic concepts and terminology that will be used and a description of how combinatorial games differ from some other types of games.

Combinatorial games

In a *combinatorial game* there are two players who take turns moving alternately. Play continues until the player whose turn it is to move has no legal moves available. No chance devices such as dice, spinners, or card deals are involved, and each player is aware of all the details of the game position (or game state) at all times. In this text, the rules of each game we study will ensure that it will end after a finite sequence of moves, and the winner is often determined on the basis of who made the last move. Under *normal play* the last player to move wins. In *misère* play the last player loses.

In fact, combinatorial game theory can be used to analyze some games that do not quite fit the above description. For instance, in DOTS & BOXES, players may make two moves in a row. Most CHECKERS positions are *loopy* and can lead to infinitely long sequences of moves. In GO and CHESS the last mover does not determine the winner. Nonetheless, combinatorial game theory has been applied to analyze positions in each of these games.

By contrast, the classical mathematical theory of games is concerned with *economic games.* In such games the players often play simultaneously and the outcome is determined by a payoff matrix. Each player's objective is to guarantee the best possible payoff against any strategy of the opponent. For a taste of economic game theory, see Problem 5.

The challenge in analyzing economic games stems from simultaneous decisions: each player must decide on a move without knowing the move choice(s) of her opponent(s). The challenge of combinatorial games stems from the sheer quantity of possible move sequences available from a given position.

Combinatorial game theory is most straightforward when we restrict our attention to *short games.* In the play of a short game, a position may never be repeated, and there are only a finite number of other positions that can be reached. We implicity (and sometimes explicitly) assume all games are short in this text.

Introducing the players

The two players of a combinatorial game are traditionally called *Left* (or just L) and *Right* (R). Various conventional rules will help you to recognize who is playing, even without a program:

Left	**Right**
Louise	Richard
Positive	Negative
bLack	White
bLue	Red
Vertical	Horizontal
Female	Male
Green	
Gray	

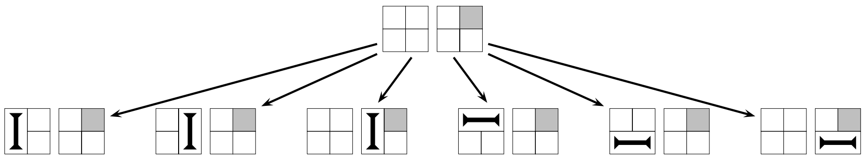
Alice and *Bob* will also make an appearance when the first player is important. To help remember all these conventions, note that despite the fact that they were introduced as long ago as the early 1980s in *WW* [BCG01], the chosen dichotomies reflect a relatively modern "politically correct" viewpoint.

Often, particularly in games involving pieces or in pen and paper games we will need a neutral color. If the game is between blue and red then this neutral color is green (because green is good for everyone!) while if it is between black and white then the neutral color is gray (because gray is neither black nor white!). Of course, this book is printed in black and white, so blue becomes black, red becomes white, and green becomes gray. That is,

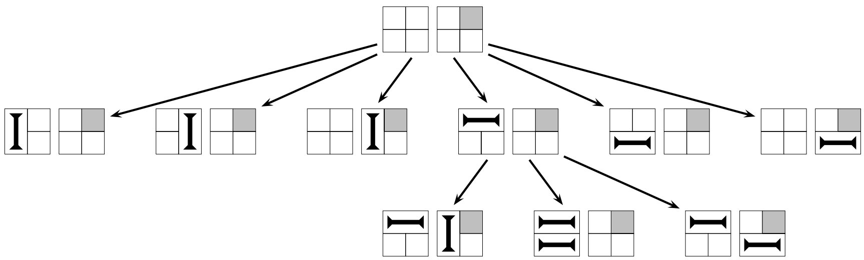
$$\begin{aligned} \text{blue} &= \text{black}, \\ \text{red} &= \text{white}, \\ \text{green} &= \text{gray}. \end{aligned}$$

Options

If a position in a combinatorial game is given and it happens to be Left's turn to move she will have the opportunity to choose from a certain set of moves determined by the rules of the game. For instance in DOMINEERING, where Left plays the vertical dominoes, she may place such a domino on any pair of vertically adjacent empty squares. The positions that arise from exercising these choices are called the *left options* of the original position. Similarly, the *right options* of a position are those which can arise after a move made by Right. The *options* of a position are simply the elements of the union of these two sets. We can draw a game tree of a position by diagrammatically listing its left and right options, with left options appearing below and to the left of the game:

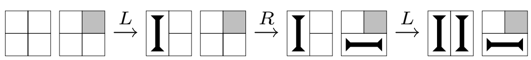


We can show as many or as few *game trees* of options as we wish:

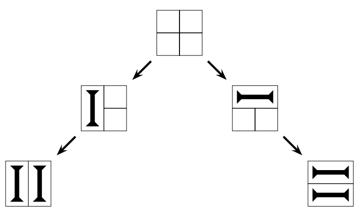


It may seem odd that we are showing two consecutive right moves in a game tree, but much of the theory of combinatorial games is based on analyzing situations where games *decompose* into several subgames. It may well be the case that in some of the subgames of such a decomposition the players do not alternate moves.

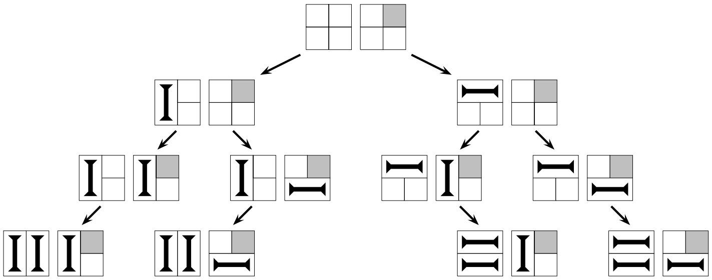
We saw this already in the DOMINEERING "square and hook" example. Left, if she wants to win, winds up making two moves in a row in the square:



Thus, we show the game tree for a square with Left and/or Right moving twice in a row:



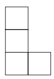
As we will see later, *dominated options* are often omitted from the game tree, when an option shown is at least as good:



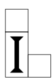
In some games the left options and the right options of a position are always the same. Such games are called *impartial*. The study of impartial combinatorial games is the oldest part of combinatorial game theory and dates back to the early twentieth century. On the other hand the more general study of non-impartial games was pioneered by John H. Conway in *ONAG* [Con01] and by Elwyn Berlekamp, John H. Conway, and Richard K. Guy in *WW* [BCG01]. Since "non-impartial" hardly trips off the tongue, and "partial" has a rather ambiguous interpretation it has become commonplace to refer to non-impartial games as *partizan games.*

To illustrate the difference between these concepts, consider a variation of DOMINEERING called CRAM. CRAM is just like DOMINEERING except that each player can play a domino in either orientation. Thus, it becomes an impartial game since there is now no distinction between legal moves for one player and legal moves for the other.

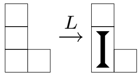
Let's look at a position in which there are only four remaining vacant squares in the shape of an L:



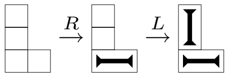
In CRAM the next player to play can force a win by playing a vertical domino at the bottom of the vertical strip, leaving



which contains only two non-adjacent empty squares and hence allows no further moves. In DOMINEERING if Left (playing vertically) is the next player, she can win in exactly this way. However, if Right is the next player his only legal move is to cover the two horizontally adjacent squares, which still leaves a move available to Left. So (assuming solid play) Left will win regardless of who plays first:



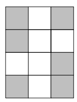
or



Much of the theory that we will discuss is devoted to finding methods to determine who will win a combinatorial game assuming sensible play by both sides. In fact, the eventual loser has no really *sensible* play[1] so a *winning strategy* in a combinatorial game is one that will guarantee a win for the player employing it no matter how his or her opponent chooses to play. Of course such a strategy is allowed to take into account the choices actually made by the opponent — to demand a uniform strategy would be far too restrictive!

Problems

1. Consider the position:



   (a) Draw the complete game trees for both CRAM and DOMINEERING. The leaves (bottoms) of the tree should all be positions in which neither player can move. If two left (or right) options are symmetrically identical, you may omit one.

   (b) Who wins at DOMINEERING if Vertical plays first? Who wins if Horizontal plays first? Who wins at CRAM?

2. Suppose that you play DOMINEERING (or CRAM) on *two* 8×8 chessboards. At your turn you can move on either chessboard (but not both!). Show that the second player can win.

3. Take the ace through five of a suit from a deck of cards and place them face up on the table. Play a game with these as follows. Players alternately pick a card and add it to the righthand end of a row. If the row ever contains a sequence of three cards in increasing order of rank (ace is low), or in decreasing order of rank, then the game ends and the player who formed that sequence is the winner. Note that the sequence need not be

[1] Unless he has some ulterior motive not directly related to the game such as trying to make it last as long as possible so that the bar closes before he has to buy the next round of drinks.

consecutive either in position or value, so for instance, if the play goes $4, 5, 2, 1$ then the $4, 2, 1$ is a decreasing sequence.

(a) Show that this is a proper combinatorial game (the main issue is to show that draws are impossible).

(b) Show that the first player can always win.

4. Start with a heap of counters. As a move from a heap of n counters, you may either:

   - assuming n is not a power of 2, remove the largest power of 2 less than n; or
   - assuming n is even, remove half the counters.

   Under normal play, who wins? How about misère play?

5. The goal of this problem is to give the reader a taste of what is *not* covered in this book. Two players play a 2×2 *zero-sum matrix game.* (Zero sum means that whatever one person loses, the other gains.) The players are shown a 2×2 matrix of positive numbers. Player A chooses a row of the matrix, and player B simultaneously chooses a column. Their choice determines one matrix entry, that being the number of dollars B must pay A. For example, suppose the matrix is

$$\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

   If player A chooses the first row with probability $1/4$, then no matter what player B's strategy is, player A is *guaranteed* to get an average of \$2.50. If, on the other hand, player B chooses the columns with 50-50 odds, then no matter what player A does, player B is *guaranteed* to have to pay an average of \$2.50. Further, neither player can guarantee a better outcome, and so B should pay player A the fair price of \$2.50 to play this game.

   In general, if the entries of the matrix game are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

   as a function of a, b, c, and d, what is the fair price which B should pay A to play? (Your answer will have several cases.)

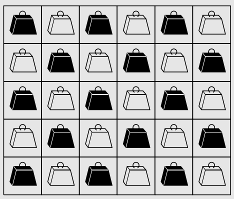
Preparation for Chapter 1

Prep Problem 1.1. Play DOTS & BOXES with a friend or classmate. The rules are found on page 267 of Appendix D. You should start with a 5×6 grid of dots. You should end up with a 4×5 grid of 20 boxes, so the game might end in a tie.

When playing a game for the first time, feel free to move quickly to familiarize yourself with the rules and to get a sense for what can happen in the game.

After a few games of DOTS & BOXES, write a few sentences describing any observations you have made about the game. Perhaps you found a juncture in the game when the nature of play changes? Did you have a strategy? (It need not be a good strategy.)

Prep Problem 1.2. Play CLOBBER with a friend or classmate. The rules are found on page 265 of Appendix D. (Note that if the Winner is not specified in a ruleset, you should assume normal play, that the last legal move wins.) You should start with a 5×6 grid of boxes:



Jot down any observation you have about the game.

Prep Problem 1.3. Play NIM (rules on page 271) with a friend or classmate. Begin with the three heap position with heaps of sizes 3, 5, and 7.

To the instructor: While DOTS & BOXES is a popular topic among students, it also takes quite a bit of time to appreciate. View the topic as optional. If you do cover it, allow time for students to play practice games. Another option is to cover it later in the term before a holiday break.

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