Linear and Non-Linear Deformations of Elastic Solids
Linear and Non-Linear Deformations of Elastic Solids

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Preface

The monograph is a comprehensive analysis of linear and nonlinear deformations in elasticity and intended not only for graduate students but for professionals in civil, mechanical, aeronautical and metallurgical engineers. The level of discussion is from elementary to the current research level. The concept of this book is divided into two sections. The first section on linear elasticity can be used as a companion text book on contact and fracture mechanics. The second section will be for those who are interested to dent into the area of nonlinear elasticity and the present venture may be the initial step for them towards this direction.

Section 1: Linear Elasticity

Elasticity is a fundamental property of all materials, building, concrete or rocks, all things in nature. Out of various topics of interest in linear elasticity viz. flexure, torsion, bending, contact and crack, etc., mainly contact and crack problems have been considered for both static and dynamic cases. The section on linear elasticity is a comprehensive work on crack and contact problems not necessarily in isotropic material, but in anisotropic as well as smart material like piezoelectric material. After introducing the stress-strain relation, the equation of motion for arbitrary time dependent body forces has been solved using Fourier and Laplace transforms. The method of solving two-dimensional singular integral equations for contact and crack problems in elliptic region has been discussed. As in any standard practice, fairly comprehensive discussion on two-dimensional contact and crack problems in isotropic material by Muskhelishvili's complex variable technique is given and in anisotropic media, Stroh's formalism. This portion is particularly suitable to graduate level engineering students who will get a glimpse of the development from the elementary level right up to research level. In a major portion of the later sections, the three-dimensional unified method valid for the Hertz contact theory and a variety of frictionless contact problem with an elliptic connection both for a rigid punch and a conical indenter has been discussed. In latter section, the elliptic crack interface between both isotropic and transversely isotropic media has been analysed. In particular, the detailed calculation of the stress intensity factor in infinite piezoelectric media has been described in detail and the fracture mechanics principle has been formulated with possible applications.

In the part of dynamical elasticity based on linear elasticity, a model of an earthquake simulation for a realistic faulting motion along an inclined geologic fault plane and a two-dimensional self-similar crack motion taking the earth as an elastic half space for simplicity has been discussed. A discussion of earthquake magnitude from the spectrum analysis is also included.

Developments on non-destructive laboratory detection of cracks in strategic defence material, ballistic missiles, aircraft, etc., require the knowledge of three-dimensional
scattering analysis. A chapter is included on scattering from an elliptic crack/inclusion, etc., by analytic method, both in low and mid-frequency range. The Wiener-Hopf method has been used to study scattering by a half-plane crack in transversely and isotropic infinite media. The study of a solid containing inclusions and distributed cracks is important in a number of engineering fields. In rock mechanics pre-existing cracks plays vital role in the optimum recovery of geothermal energy, oil and gas. A section containing the theory of effective moduli of composite has been included both by static and dynamic method. In static cases, besides the Mori Tanaka method, the Kuster-Toksöz model has been used in rock mechanics. The basic idea about a numerical method like BEM which is of particular interest to the engineering community has also been included.

**Section II: Nonlinear Elasticity**

Any physical system is nonlinear in general. Real systems involve randomness or stochastic behaviour. Thus, a natural system may be stochastic as well as nonlinear.

In designing structures for construction of bridges, aircraft, missile, hydrospace, shipbuilding, transportation and high-rise buildings, the small deflection theory cannot satisfy the requirements of the design engineers. High speed aeroplanes, missiles and space vehicles are often subjected to large deflections and reveal nonlinear response. But the large deflection theory involves nonlinear equations which are not easy to solve analytically because of its complex nature. To model the problem with its inherent nonlinearities and random fluctuations or uncertain data, some new techniques need to be used.

Section II of the monograph contains seventeen chapters devoted to nonlinear equations of elasticity and their applications to physical problems. Nonlinear vibration of beams, large deflection of ordinary and sandwich plates of different shapes, large deflection of ordinary and sandwich cylindrical shells, orthotropic cylindrical shells, and shells with variable thickness are discussed because of their practical importance. In the above-mentioned cases, vibration and stability of structures have also been discussed for necessity in design and construction. Since thermal effect plays an important role in practical situations, heated structures have also been considered for analysis. Because it is not always possible to find analytical solutions of the nonlinear problems, approximate methods like Ritz’s method, Galerkin’s method, Berger’s method, Banerjee’s modified method and Mazumdar’s method modified by Bera have been used for obtaining the nonlinear solutions. The Adomian Decomposition Method (ADM), which does not require modification like the perturbation method, has also been applied to find the solution in case of orthotropic sandwich shell.

In writing this book some errors are likely to crop up in the above works. The authors will greatly appreciate if the readers bring these errors to their notices.
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Section I

Linear Elasticity
1

Basic Fundamentals and an Overview

1.1 Introduction

We introduce a basic stress system in an elastic media. Details can be found in any standard text book on elasticity (Love, 1944, Green and Zerna, 1960). We derive the body force equivalent for the body force term in the equation of motion corresponding to displacement and stress discontinuity across a surface. We also derive the representation theorem in terms of Green's function and derivatives using reciprocity theorem. Following Roy (1979) the transform method is used to solve the equation of motion for arbitrary body force. In the process, we derive the Green’s function in an infinite and semi-infinite medium. The last section includes a discussion on basic principle of fracture mechanics.

1.2 Basic Stress System

Under external loading on the boundary of an elastic body, a material point $P$ inside $V$ is displaced from the equilibrium position of rest to a neighbouring point. A planar surface having a normal $n$, at $P$ experiences a stress $\tau_{ij}$ also designated as $\sigma_{ij}$. In linear elasticity one is concerned with infinitesimal deformation. All quantities like displacement $u_i$, etc., are piecewise differentiable functions of coordinate $x_i$ and time $t$. The deformation vector is

$$d_i = u_i(x_i + dx_i, t) - u_i(x_i, t) \approx \frac{\partial u_i}{\partial x_j} dx_j$$

(1.1)

neglecting the higher derivatives since the deformation is infinitesimal.

The quantities $\frac{\partial u_i}{\partial x_j}$ constitute a Cartesian Tensor of second rank which can be easily seen when changing over to a new coordinate system such as $x'_i = l_{ij} x_j$.

One then obtains the symmetric and antisymmetric tensor $e_{ij}$ and $\omega_{ij}$, where

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x'_j} + \frac{\partial u_j}{\partial x'_i} \right), \quad \omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x'_j} - \frac{\partial u_j}{\partial x'_i} \right)$$

(1.2)
designated as strain and rotation tensors respectively and

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2} \left( e_{ij} + \omega_{ij} \right)$$

The strain tensor so defined is symmetric (i.e., $e_{ij} = e_{ji}, e_{11}, e_{22}, e_{33}$) are the longitudinal strains along the $x_i, (i = 1, 2, 3)$ axes. $e_{ij}, (i \neq j)$ are the shear strain. They satisfy various compatibility conditions.

The stress-like strain is a tensor of the second order. By Hook’s law they are linearly related to the strain tensor in linear elasticity. In general, in anisotropic elastic media we have

$$\tau_{ij} = C_{ijkl} e_{kl}$$  \hspace{1cm} (1.3)

where $C_{ijkl}$ (in all 81) are called elastic moduli, assumed constants in general. However, they may be considered a function of the coordinate system as in inhomogeneous medium (e.g., in functionally graded media). Symmetry reduces 81 constants to 36. Further, if a strain energy function exists the number of independent constants is 21.

In the isotropic medium elastic properties are the same in all directions. In this case there are only two independent elastic moduli, $\lambda$ and $\mu$, called Lame’s elastic constants.

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$  \hspace{1cm} (1.4)

$$\tau_{ij} = \lambda e_{ij} \delta_{ij} + 2\mu e_{ij}$$

Other parameters that will be required are the Young’s modulus $E$, Poisson’s ratio $\nu$. They are in terms of $\lambda$ and $\mu$

$$E = \frac{\mu (3\lambda + 2\mu)}{\lambda + \mu}, \hspace{0.5cm} \nu = \frac{\lambda}{\lambda + \mu}$$  \hspace{1cm} (1.5)

In a transversely isotropic medium (i.e., a medium having hexagonal symmetry), the stress system is given in terms of five parameters $c_{ij}$

$$\sigma_{xx} = c_{11} \frac{\partial u}{\partial x} + (c_{11} - 2c_{66}) \frac{\partial v}{\partial y} + c_{13} \frac{\partial w}{\partial z}$$

$$\sigma_{yy} = (c_{11} - 2c_{66}) \frac{\partial u}{\partial x} + c_{11} \frac{\partial v}{\partial y} + c_{13} \frac{\partial w}{\partial z}$$

$$\sigma_{zz} = c_{13} \frac{\partial u}{\partial x} + c_{13} \frac{\partial v}{\partial y} + c_{33} \frac{\partial w}{\partial z}, \hspace{0.5cm} \tau_{xy} = c_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\tau_{yz} = c_{44} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right), \hspace{0.5cm} \tau_{zx} = c_{44} \left( \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x} \right)$$  \hspace{1cm} (1.6)
1.3 Equation of Motion and Various Potentials

The principle of linear balance of linear momentum for the motion of particles under body forces $f_i$ in an arbitrary volume $V$ yields the basic equation of motion as

$$\tau_{ij,j} + f_i = \rho \ddot{u}_i$$

Written in full,

$$\frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial u_k}{\partial x_l} \right) + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$ (1.7)

In an isotropic medium, the equation of motion is

$$\nabla (\lambda + \mu) (\nabla . u) + 2\mu \nabla^2 u = \rho \ddot{u}$$

$$\frac{\partial}{\partial x_j} \left( (\lambda + \mu) \frac{\partial u}{\partial x_i} + 2\mu \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial x_k^2} \right) \right) u_i + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$ (1.8)

We now discuss some general methods of solving the equation of motion. One method is to find potential $\phi, \psi, \chi$ such that

$$u = \text{grad} \phi + \text{div} \text{grad} \psi + \text{curl} \text{curl} \chi = \nabla \phi + \nabla \nabla \psi + \nabla \times \nabla \chi$$ (1.9)

Substituting Eq. (1.9) in Eq. (1.8) we find, potential $\phi, \psi, \chi$ the wave equation

$$\nabla^2 (\phi, \psi, \chi) = \left( \frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\beta^2} \right) \frac{\partial^2}{\partial t^2} (\phi, \psi, \chi)$$ (1.10)

$\phi$ is the longitudinal wave potential with velocity, $\alpha^2 = (\lambda + 2\mu) / \rho$ and $\psi$ is the SV wave potential with $\beta^2 = \frac{\mu}{\rho}$ and $\chi$ is the SH wave potential with the velocity $\beta$.

In the elastostatic case, the potentials are (Youngdahl, 1989)

$$u = \nabla \psi - z \nabla \psi + (3 - 4\nu) k \psi + \nabla \times (k \chi)$$ (1.11)

where each of $(\phi, \psi, \chi)$ satisfy the harmonic equation

$$\nabla^2 (\phi, \psi, \chi) = 0, \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ (1.12)
1.4 Various Transforms Used

We introduce various transforms for solving the equation of motion. While the potential method via the individual wave equation solution is one approach, we usually follow the transform method. We introduce the Fourier transform

\[ f(k) = \int_{-\infty}^{\infty} f(x) e^{i\xi x} \, dx, \]

with the inverse transform defined by

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k) e^{-i\xi x} \, dk \tag{1.13} \]

Similarly, the Fourier transform of functions in two variables is

\[ f(\xi, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{i(\xi x + \eta y)} \, dx \, dy, \]

\[ f(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i(\xi x + \eta y)} \, dx \, dy, \tag{1.14} \]

Hankel’s transform of \( n \)-th order and its inverse are defined by

\[ f(k) = \int_{0}^{\infty} f(r) J_n(kr) \, dr \]

\[ f(r) = \int_{0}^{\infty} f(k) J_n(kr) \, dk \tag{1.15} \]

We note further that we represent \( \delta(x) = \delta(x_1) \delta(x_2) \delta(x_3) \) and \( H(t) \) as the Dirac’s delta function and Heaviside unit function.

1.5 General Form of the Elastic Wave Equation

The transform method is used to obtain the solution of the elastic wave equation for arbitrary body forces. We note, the Earth, to a great extent, behaves as an elastic medium. Basic equations of motion are

\[ \rho \frac{\partial^2 u_i}{\partial t^2} = \rho F_i + \sum_{ijkl} c_{ijkl} \frac{\partial}{\partial x_j} \left( \frac{\partial u_k}{\partial x_l} \right) \tag{1.16} \]

It is through the body force term \( F_i \) that the simulation of the earthquake process is introduced in the mathematic model. The general type of \( F_i \) considered in an earthquake source mechanism study is either a point source \( F = f \delta(r - r_0) g(t) \) or a couple source. The general form of a single couple is \( F = Mn \nabla f \delta(r - r_0) g(t) \), where \( \nabla \) is the gradient vector, \( f \) is the direction of \( F \), \( M \) is the moment, and \( n \) is the arm of the couple.
A double couple is represented by

$$F = M [n \nabla f + f \nabla n] \delta (r - r_i) g (t)$$

(1.17)

The other most popular model is to consider the earthquake resulting from crack initiation and propagation. A crack results in a discontinuity in the displacement across the crack faces and is thus a dislocation model. The dynamics of the crack source depends on the release of the stress at source region and is governed by the fracture criterion.

Let the displacement or stress be discontinuous in the source coordinate system \((\zeta_1, \zeta_2, \zeta_3)\) across the plane \(\zeta_3 = \zeta_{30}\). To obtain the derivatives in a dislocation model we regard the derivatives as the generalized derivative. Thus, the generalized derivative of \(u_i (\zeta_1, \zeta_2, \zeta_3)\) be represented as \(\frac{\partial}{\partial \zeta_j} (u_i)\). Then from Jones (1964) we have

$$\frac{\partial}{\partial \zeta_i} (u_i) = \frac{\partial u_i}{\partial \zeta_j} + [u_i] \delta (\zeta_3 - \zeta_{30})$$

$$\frac{\partial^2}{\partial \zeta^2} (u_i) = \frac{\partial^2 u_i}{\partial \zeta^2} + \frac{\partial [u_i]}{\partial \zeta_3} \delta (\zeta_3 - \zeta_{30}) + [u_i] \delta (\zeta_3 - \zeta_{30})$$

(1.18)

$$\frac{\partial^2}{\partial \zeta^2} (u_i) = \frac{\partial^2 u_i}{\partial \zeta^2} \quad (j = 1, 2)$$

where \([u_i] = u_i^+ - u_i^-\) is the jump across \(\zeta_3 = \zeta_{30}\).

Substituting the relation Eq. (1.18) in the equation of motion (1.13) where we assume the derivatives as generalized derivative, we obtain for the isotropic case the equivalent body force as

$$F_1 = -[\tau_{13}] \delta (\zeta_3 - \zeta_{30}) - \lambda \left[ \frac{\partial u_1}{\partial \zeta_1} \right] \delta (\zeta_3 - \zeta_{30}) - \mu [u_1] \delta' (\zeta_3 - \zeta_{30})$$

$$F_2 = -[\tau_{23}] \delta (\zeta_3 - \zeta_{30}) - \lambda \left[ \frac{\partial u_2}{\partial \zeta_2} \right] \delta (\zeta_3 - \zeta_{30}) - \mu [u_2] \delta' (\zeta_3 - \zeta_{30})$$

$$F_3 = -[\tau_{33}] \delta (\zeta_3 - \zeta_{30}) - \mu \left[ \frac{\partial u_1}{\partial \zeta_1} + \frac{\partial u_2}{\partial \zeta_2} \right] \delta (\zeta_3 - \zeta_{30}) -$$

$$- (\lambda + 2\mu) [u_3] \delta' (\zeta_3 - \zeta_{30})$$

(1.19)

where \([\tau_{ij}]\) and \([u_i]\) are the jump in the stress and displacement across \(\zeta_3 = \zeta_{30}\).

### 1.6 Reciprocity Principle and Representation Theorem

The reciprocity principle, which is an important tool in the study of wave propagation in various fields (e.g., elastodynamic, electrodynamic, etc.), follows on writing the transformed equation of motion in the operator form, namely
Linear and Non-Linear Deformations of Elastic Solids

\[ L u_t + f_i = -\rho \omega^2 u_i, \quad \forall x \in V \]  (1.20)

where

\[ L u_t = \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} \left( c_{ijpq} \frac{\partial u_p}{\partial x_q} \right) \]

Let \( v_i = v_i(x, \omega) \) be an alternate displacement field which satisfies the conjugate differential equation associated with force \( g_i \),

\[ L^* v_j + g_j = -\rho \omega^2 v_j, \quad \forall x \in V \]  (1.21)

where

\[ L^* v_i = \frac{\partial}{\partial x_j} \left( c_{ijpq} \frac{\partial v_p}{\partial x_q} \right) = L v_i. \]

Since \( L \) is self–adjoint and \( c_{ijpq} \) is symmetric, \( L = L^* \). Now multiplying Eq. (1.20) by \( v_j \) and Eq. (1.21) by \( u_i \) and integrating over the volume \( V \) we obtain

\[ \int_V (v_j L u_t - u_i L^* v_j) dV = \int_V (v_j f_i - u_i g_j) dV \]

On transforming the volume integral \( V \) to the surface integral using Gauss integral we obtain in view of the Hermitian property of \( L \),

\[ \int_s (v_j \tau_{ij} - u_i \tau_{ji}) n_j dS = \int_V (v_j f_i - u_i g_j) dV \]  (1.22)

In deriving Eq. (1.22) we have not imposed any boundary conditions on the surface \( S \).

Let us introduce Green’s function or rather Green’s tensor \( G_{ij}(x, \xi, \omega) \). \( G_{ij} \) satisfies the equation

\[ \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial G_{lm}}{\partial x_l} \right) + \rho \omega^2 G_{lm} = -\delta_{lm} \delta \left(x - \xi \right) \]  (1.23)

Thus \( G_{ij}(x, \xi, \omega) \) is defined as the time transformed displacement at \( x \) in the \( i \) direction associated with a point force in the \( j \) direction.

Let now assume the stress on the bounding surface \( S \) vanish. Choose

\[ f_i = \delta_{im} \delta \left(x - \xi \right), \quad g_i = \delta_{im} \delta \left(x - \xi' \right) \]  (1.24)

with the corresponding displacement \( u_i \) and \( v_i \) be designated as Green’s tensor \( G_{im}(x, x', \omega) \) and \( G_{im} \) respectively.
Then substituting the values of \( f_i, g_j \) Eq. (1.24) in Eq. (1.22) along with the vanishing stress on \( S \) and the integral

\[
\int_V \delta_{ji} \delta_{ip} \delta_j (x' - x) G_{ip}(x, \xi, \omega) dV = G_{ji}(x', \xi, \omega)
\]

we obtain \( G_j(\xi, \xi', \omega) = G_{ji}(\xi', \xi, \omega) \)

or in general \( G_j(\xi, \xi, \omega) = G_{ji}(\xi, x, \omega) \) (1.25)

The reciprocity relation (1.25) is now used to obtain the representation theorem. Let us choose \( g_i(x, \omega) = \delta_{in} \delta(x - \xi) g(\omega) \)

and substitute in Eq. (1.22). We choose the corresponding displacement and traction as given below:

\[
v_i(x, \xi, \omega) = G_{in}(x, \xi, \omega), \quad T_i^n(x, \xi, \omega) = c_{ijkl} \frac{\partial v_k}{\partial x_l} n_j = c_{ijkl} \frac{\partial G_{in}}{\partial x_l} n_j
\]

and finally the representation theorem is obtained as

\[
u_n(x, \omega) = \int_V G_{in}(x, \xi, \omega) f_i(x, \xi, \omega) dV_x + \int_S \left[ G_{in}(x, \xi, \omega) \tau_j(x, \omega) - u_i(x, \omega) c_{ijkl} \frac{\partial G_{in}}{\partial x_l} \right] n_j dS \tag{1.26}
\]

The corresponding representation theorem in time domain, on taking the inverse transform is

\[
u_n(x, t) = \int_V G_{in}(x, \xi, t - \tau) f_i(x, \xi, \tau) dV_x = \int_S \left[ G_{in}(x, \xi, t - \tau) \tau_j(x, \tau) - u_i(x, \tau) c_{ijkl} \frac{\partial G_{in}}{\partial x_l} \right] n_j dS \tag{1.27}
\]

If there is a stress discontinuity or dislocation across an internal surface \( \Sigma \) there will be an additional integration over the internal surface. If further the surface \( S \) is at infinity the contribution of the integral over \( S \) vanishes due to the radiation condition at infinity and only the integral over \( \Sigma \) remains. Finally, we have

\[
u_n(x, t) = \int_{-\infty}^{\infty} d\tau G_{in}(x, \xi, t - \tau) f_i(x, \xi, \tau) dV_x - \int_{-\infty}^{\infty} d\tau \int_{\Sigma} G_{in}(x, \xi, t - \tau) \left[ \tau_j(x, \tau) v_j \right] + \left[ u_i(x, \tau) \right] c_{ijkl} \frac{\partial G_{in}}{\partial x_l} v_j d\Sigma \tag{1.28}
\]

where, \( \left[ \tau_j(x, \tau) \right] = \tau_j^+ - \tau_j^-, \quad \left[ u_i(x, \tau) \right] = u_i^+ - u_i^- \)

are the discontinuity across the surface \( \Sigma \).
1.7 General Solution of the Equation of Motion for an Arbitrary Force System

Let an orthogonal Cartesian coordinate system \((\zeta_1, \zeta_2, \zeta_3)\), called the source coordinate system, describe the source system in the medium and another system on orthogonal Cartesian coordinate system \((x, y, z)\) describe the elastic half-space with \(z = -h\) be the free surface. The origin is at the centre of the circular source and \(xy\)-plane parallel to the free surface. Let \(\zeta_3 = 0\) be the inclined fault plane, in our case. The two coordinate systems are related by

\[
(x, y, z)^T = A_0 (\zeta_1, \zeta_2, \zeta_3)^T
\]

(1.29)

where \(A_0\) is the orthogonal matrix and \(T\) denotes the transpose of the matrix.

Similarly the displacements in \((x, y, z)\)-system \(u = (u_1, u_2, u_3)\) and \((\zeta_1, \zeta_2, \zeta_3)\)-system \(u' = (u'_1, u'_2, u'_3)\) are related by \(u = A_0 u'\). Here \(a_ia_0 = \delta_{ij}\) where \(\delta_{ij}\) is the Kronecker delta function, the equation of motion in an infinite medium is now written in the source coordinate system as

\[
\rho \frac{\partial^2 u'_i}{\partial t^2} = \sigma_{ij} + F'_i
\]

(1.30)

where \(F'_i\) \((i = 1, 2, 3)\) vanishes outside the source region. \(F'_i\) may be a planar, point or volume source and

\[
\sigma_{ij} = \lambda \nabla \cdot u + \mu \left( \frac{\partial u'_i}{\partial \zeta_j} + \frac{\partial u'_j}{\partial \zeta_i} \right)
\]

(1.31)

where Kronecker delta \(\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}\)

\(\lambda, \mu\) and \(\rho\) are elastic moduli and density respectively.

We introduce the following Fourier and Laplace transforms respectively as

\[
U_j(k, v, s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_j(\zeta_1, \zeta_2, \zeta_3) e^{-i(k \zeta_1 + v \zeta_2 + s \zeta_3)} d\zeta_1 d\zeta_2 d\zeta_3
\]

\[
F(p) = \int_0^\infty f(t) e^{-pt} dt
\]

(1.32)

Transforming Eq. (1.20) we obtain
\[
\left(\beta^2 s^2 + \alpha^2 k^2 + \beta^2 v^2 + p^2\right)U''_1(k,v,s,p) + \left(\alpha^2 - \beta^2\right)kvU'_1(k,v,s,p)
\]
\[
+ \left(\alpha^2 - \beta^2\right)\frac{F'_1(k,v,s,p)}{\rho}(\alpha^2 - \beta^2)
\]
\[
kvU''_1(k,v,s,p) + \left(\beta^2 s^2 + \alpha^2 v^2 + \beta^2 k^2 + p^2\right)U'_1(k,v,s,p)
\]
\[
+ \left(\alpha^2 - \beta^2\right)\frac{F'_1(k,v,s,p)}{\rho}(\alpha^2 - \beta^2)
\]
\[
ksU''_1(k,v,s,p) + \left(\alpha^2 - \beta^2\right)vsU'_1(k,v,s,p)
\]
\[
+ \left(\alpha^2 s^2 + \beta^2 k^2 + \beta^2 v^2 + p^2\right)U'_1(k,v,s,p) = \frac{F'_1(k,v,s,p)}{\rho}
\]

(1.33)

Solving Eq. (1.33) for \(U'_j\) (for \(j = 1, 2, 3\)) and taking the inverse Fourier and Laplace transforms successively we obtain the transformed variable

\[
u'_j = \frac{1}{(2\pi)^4 i} \int_{Br} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{p \tau + ik_1 + \rho \zeta_1 + \rho \zeta_2 + \rho \zeta_3} dk dv ds dp
\]

(1.34)

where

\[
s_\alpha = \left(k^2 + v^2 + \frac{p^2}{\alpha^2}\right)^{1/2}, \quad s_\beta = \left(k^2 + v^2 + \frac{p^2}{\beta^2}\right)^{1/2}
\]

(1.35)

and \(Br\) denotes the Bromwich contour. \(P_j\) can be easily written from Eq. (1.5) in terms of adjoint matrices of \((3 \times 3)\) matrix formed with coefficients of \(U_J\) in the left-hand side of Eq. (1.5).

We note that \(u'_j\) are the displacement components along \((\zeta_1, \zeta_2, \zeta_3)\) and are related through \(u = Au'\) with the displacement components \(u_j\) along \((x,y,z)-\)axes. We now change over to the coordinate system \((x,y,z)\) from \((\zeta_1, \zeta_2, \zeta_3)\). The integration variable \((k,v,s)\) are now related to the new integration variable \((\xi, \eta, \zeta)\) as

\[
\Omega = A_k \Omega
\]

(1.36)

or

\[
\Omega' = A_k^{-1} \Omega
\]

(1.37)

where \(\Omega\) and \(\Omega'\) are the column matrices with element \((k,v,s)\) and \((\xi,\eta,\zeta)\) respectively.
Thus, on changing over to the new integration variables, the displacements are given by

\[
u_i = \frac{1}{(2\pi)^4} \int B \int \int \int \int \int \frac{P_j e^{pt+i(\xi x + \eta y + \zeta z)}}{\alpha^2 \beta^4 (\zeta^2 + \nu^2_\alpha) (\zeta^2 + \nu^2_\beta)} dp d\xi d\eta d\zeta (1.38)
\]

where

\[
\nu_\alpha = \left( \xi^2 + \eta^2 + \frac{p^2}{\alpha^2} \right)^{1/2}, \quad \nu_\beta = \left( \xi^2 + \eta^2 + \frac{p^2}{\beta^2} \right)^{1/2} (1.39)
\]

and \(P_j\) can be written from \(P_j\) on making use of transformation rule (1.8).

We now evaluate Eq. (1.31). Let us assume that the body forces are distributed over a region extending from \(z = 0\) to \(z = \h_0\). We note that the only poles in the \(\zeta\)-plane are \(\zeta = \pm \nu_\alpha\) and double poles at \(\zeta = \pm \nu_\beta\). Then on evaluating for \(z < 0\) at the respective poles (i.e., at \(\zeta = -\nu_\alpha, \zeta = \nu_\beta\)) the displacement in an infinite medium, after simplification, can be written in \((x,y,z)\) system as, for \(z < 0\)

\[
u_{\text{inc}} = \int B \int \int \int \int \int \frac{e^{pt+i(\xi x + \eta y + \zeta z)}}{16\pi^2 p^3 i \rho} \left[ i \left\{ (\xi + \eta j) - \zeta k \right\} \frac{A}{\zeta_\alpha} e^{\nu_\alpha z} + B e^{\nu_\beta z} \right.
\]

\[
\times \left\{ i (\xi + \eta j) + \frac{\xi^2 + \eta^2}{\zeta_\beta} \right\} + \frac{p^2}{\beta^2} C (\eta i - \xi j) e^{\nu_\beta z} \right] d\xi d\eta d\rho (1.40)
\]

where \(\nu_{\text{inc}}\) denotes the incident field; \(i,j,k\) are unit vectors in the directions of \(x-, y-, z\)-axes and

\[
A = i \left( D_\alpha \xi + E_\alpha \eta \right) + \nu_\alpha G_\alpha
\]

\[
B = i \left( D_\beta \xi + E_\beta \eta \right) - \frac{\nu_\beta G_\alpha}{\xi^2 + \eta^2} + G_\alpha
\]

\[
C = \frac{i \left( D_\beta \eta - E_\beta \xi \right)}{\left( \xi^2 + \eta^2 \right) \nu_\beta} (1.41)
\]

\[
D = F_1(k,v,s,p) a_{11} + F_2(k,v,s,p) a_{12} + F_3(k,v,s,p) a_{13}
\]

\[
E = F_1(k,v,s,p) a_{21} + F_2(k,v,s,p) a_{22} + F_3(k,v,s,p) a_{23}
\]

\[
G = F_1(k,v,s,p) a_{31} + F_2(k,v,s,p) a_{32} + F_3(k,v,s,p) a_{33}
\]

\(D_\alpha\) and \(D_\beta\) are obtained from \(D\) after changing over from variable \((k,v,s)\) to \((\xi,\eta,\zeta)\) through transformation (1.8) and setting \(\zeta = -\nu_\alpha, \zeta = -\nu_\beta\) respectively. Similar meaning is attached to \(E_\alpha, E_\beta, G_\alpha, G_\beta\).
In particular, when the source coordinate \((\zeta_1, \zeta_2, \zeta_3)\) are the same as \((x, y, z)\) coordinate system, we have

\[
D = X(\zeta, \eta, \zeta, p), \quad E = Y(\zeta, \eta, \zeta, p), \quad F = Z(\zeta, \eta, \zeta, p)
\]

To obtain the displacement field in the elastic half space, we add up to the incident field in \(-h < z < 0\) given by Eq. (1.9) the reflected field, \(u_{\text{ref}}\) which is the solution of the equation of motion. The corresponding potentials \(\phi, \Psi_1, \chi_i\) are respectively solution of appropriate wave equation and are given by

\[
u_{\text{ref}} = \nabla \phi + \nabla \times (\Psi_i k) + \nabla \times (\chi k) \tag{1.42}
\]

where

\[
(\phi, \Psi_1, \chi_i) = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{i(\xi + \eta y)}}{16\pi^2 i} \left[ A'_i(\xi, \eta) e^{-\nu z}, B'_i(\xi, \eta) e^{-\psi z}, C'_i(\xi, \eta) e^{-\chi z} \right] d\xi d\eta dp
\]

The constants \(A'_i, B'_i, C'_i\) are obtained from the condition of vanishing of stress on the free surface \(z = -h\) (i.e., \(\sigma_{ij}^{\text{inc}} + \sigma_{ij}^{\text{ref}} = 0\)).

Then the values of \(u_{\text{ref}}\) is given by

\[
u_{\text{ref}} = \int_{-b}^b \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{i(\xi + \eta y)}}{16\pi^2 i} \left[ A' \nu_\alpha(\xi, \eta) - \frac{4B F(\xi, \eta)}{v_\alpha F(\xi, \eta)} \left(2\xi^2 + 2\eta^2 + \frac{p^2}{\beta^2}\right) \right] d\xi d\eta dp
\]

In particular the surface displacement \(u(t, x, y, -h) = u_{\text{inc}} + u_{\text{ref}}\) can be written on setting \(z = -h\) as

\[
u(t, x, y, -h) = \frac{1}{8\pi^2 i} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{i(\xi + \eta y)}}{\mu F(\xi, \eta)} \left[ A e^{-\nu p} \left(2v_\mu i(\xi i + \eta j) - \left(2\xi^2 + 2\eta^2 + \frac{p^2}{\beta^2}\right) \right) \right]
\]

In particular

\[
u(t, x, y, -h) = \frac{1}{8\pi^2 i} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{i(\xi + \eta y)}}{\mu F(\xi, \eta)} \left[ A e^{-\nu p} \left(2v_\mu i(\xi i + \eta j) - \left(2\xi^2 + 2\eta^2 + \frac{p^2}{\beta^2}\right) \right) \right]
\]

\[
u(t, x, y, -h) = \frac{1}{8\pi^2 i} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{i(\xi + \eta y)}}{\mu F(\xi, \eta)} \left[ A e^{-\nu p} \left(2v_\mu i(\xi i + \eta j) - \left(2\xi^2 + 2\eta^2 + \frac{p^2}{\beta^2}\right) \right) \right]
\]

\[
u(t, x, y, -h) = \frac{1}{8\pi^2 i} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{i(\xi + \eta y)}}{\mu F(\xi, \eta)} \left[ A e^{-\nu p} \left(2v_\mu i(\xi i + \eta j) - \left(2\xi^2 + 2\eta^2 + \frac{p^2}{\beta^2}\right) \right) \right]
\]

\[
u(t, x, y, -h) = \frac{1}{8\pi^2 i} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{i(\xi + \eta y)}}{\mu F(\xi, \eta)} \left[ A e^{-\nu p} \left(2v_\mu i(\xi i + \eta j) - \left(2\xi^2 + 2\eta^2 + \frac{p^2}{\beta^2}\right) \right) \right]
\]

\[
u(t, x, y, -h) = \frac{1}{8\pi^2 i} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{i(\xi + \eta y)}}{\mu F(\xi, \eta)} \left[ A e^{-\nu p} \left(2v_\mu i(\xi i + \eta j) - \left(2\xi^2 + 2\eta^2 + \frac{p^2}{\beta^2}\right) \right) \right]
\]

\[
u(t, x, y, -h) = \frac{1}{8\pi^2 i} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{i(\xi + \eta y)}}{\mu F(\xi, \eta)} \left[ A e^{-\nu p} \left(2v_\mu i(\xi i + \eta j) - \left(2\xi^2 + 2\eta^2 + \frac{p^2}{\beta^2}\right) \right) \right]
\]

\[
u(t, x, y, -h) = \frac{1}{8\pi^2 i} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{i(\xi + \eta y)}}{\mu F(\xi, \eta)} \left[ A e^{-\nu p} \left(2v_\mu i(\xi i + \eta j) - \left(2\xi^2 + 2\eta^2 + \frac{p^2}{\beta^2}\right) \right) \right]
\]

\[
u(t, x, y, -h) = \frac{1}{8\pi^2 i} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{i(\xi + \eta y)}}{\mu F(\xi, \eta)} \left[ A e^{-\nu p} \left(2v_\mu i(\xi i + \eta j) - \left(2\xi^2 + 2\eta^2 + \frac{p^2}{\beta^2}\right) \right) \right]
\]

\[
u(t, x, y, -h) = \frac{1}{8\pi^2 i} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{i(\xi + \eta y)}}{\mu F(\xi, \eta)} \left[ A e^{-\nu p} \left(2v_\mu i(\xi i + \eta j) - \left(2\xi^2 + 2\eta^2 + \frac{p^2}{\beta^2}\right) \right) \right]
\]
where

\[
F(\xi, \eta) = \left( 2\xi^2 + 2\eta^2 + \frac{p^2}{\beta^2} \right)^2 - 4(\xi^2 + \eta^2)\nu_\alpha\nu_\beta
\]  
(1.44)

and \(A, B, C\) are given by Eq. (1.38). The results are valid for any arbitrary force system.

### 1.8 Green’s Function in an Infinite Medium

We note that Green’s function \(G_{ij}\) is the \(i\)-th component of displacement for a point force in the \(j\)-direction. We have for a point source in the \(i\)-direction

\[
\bar{F}_j = \int_0^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\xi x-i\eta y} e^{-i\xi z} e^{-i\xi x-i\eta y} e^{-i\xi z} \delta(x-x_0) \delta(y-y_0) \delta(z-z_0) \, dx \, dy \, dz \\
\times dx dy dz dt \\
= \delta(p + i\omega)e^{-i\xi x_0-i\eta y_0-i\xi z_0}
\]  
(1.45)

Also, we have the relation

\[
\frac{1}{2\pi} \int_0^\infty \int_{-\infty}^{\infty} \int \frac{1}{\nu} e^{-i\xi(x-x_0)-i\eta(y-y_0)-i\xi z} \, dx \, dy \, dz = \frac{e^{-i\omega R}}{R}
\]  
(1.46)

and

\[
v = \sqrt{\xi^2 + \eta^2 - \omega^2/c^2}, \quad R = \left[ (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \right]^{1/2}
\]

Substituting these values in Eq. (1.40) and computing the displacement \(u_t\) in the \(x\) direction for successive value \(j = 1, 2, 3\) one generates

\[
(H_{11}, H_{12}, H_{13}) = \frac{e^{-i\omega t}}{4\pi\rho \omega^2} \left( f_1 \frac{\partial^2}{\partial x_1^2}, f_2 \frac{\partial^2}{\partial x_2^2}, f_3 \frac{\partial^2}{\partial x_3^2} \right) \left( \frac{e^{i\omega R}}{R} - \frac{e^{i\alpha R}}{R} \right)
\]

\[
+ \left( f_1, f_2, f_3 \right) \frac{\omega^2}{\beta^2} \frac{e^{i\omega R}}{R}
\]

Hence, on setting \(f_1 = f_2 = f_3 \equiv 1\) on obtaining similarly the other displacement component and identifying \(H_{ij}\) as the Green’s tensor \(G_{ij}\), we obtain

\[
G_{ij} = \frac{e^{-i\omega t}}{4\pi\rho \omega^2} \left[ \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{e^{i\omega R}}{R} - \frac{e^{i\alpha R}}{R} \right) + \frac{\omega^2}{\beta^2} \delta_{ij} \frac{e^{i\omega R}}{R} \right]
\]  
(1.47)
On taking the limit of Eq. (1.47) as \( \omega \to 0 \), elastostatic Green’s tensor can be found as,

\[
G_{ij}^0 = \frac{1}{4\pi\mu} \left[ \frac{1}{1-v} \frac{\partial^2}{\partial x_i \partial y_j} R + \frac{\delta_{ij}}{R} \right]
\]

(1.48)

To obtain the Green’s tensor in two dimensions we use the result

\[
I = \int_{-\infty}^{\infty} e^{-i\omega R} \frac{dz}{R} = 2\int_{-\infty}^{\infty} e^{-i\omega \rho \sqrt{p^2 - r^2}} dp = 2iH_0^1 \left( \frac{\omega r}{c} \right)
\]

(1.49)

on substituting \( R^2 = z^2 + r^2 = p^2 \),

where \( H_0^1(z) \) is the Hankel function of the first kind.

Thus, the two-dimensional Green’s tensor is

\[
G^t_{ij} = \frac{ie^{-i\omega t}}{2\rho \omega^2} \left[ \frac{\partial^2}{\partial x_i \partial x_j} \left\{ H_0^1 \left( \frac{\omega r}{\beta} \right) - H_0^1 \left( \frac{\omega r}{\alpha} \right) \right\} + \frac{\omega^2}{\beta^2} \delta_{ij} H_0^1 \left( \frac{\omega r}{\beta} \right) \right]
\]

(1.50)

where

\[
r = \left[ (x - \xi_0)^2 + (y - \eta_0)^2 \right]^{1/2}
\]

For arbitrary time dependent force \( f(t) \), Green’s tensor in infinite medium is obtained on multiplying by the Fourier transform of \( f(\omega) \) of \( f(t) \) as

\[
G_{ij}(x,\xi,t) = \int_{-\infty}^{\infty} G_{ij}(x,\xi,\omega) f(\omega) e^{-i\omega t} d\omega =
\]

\[
\frac{1}{4\pi\rho} \int_R^\infty \left[ \frac{3\gamma_i\gamma_j - \delta_{ij}}{R^3} \int_0^\infty \frac{R}{\alpha} \tau f(t-\tau) d\tau + \frac{1}{R^2} \gamma_i\gamma_j f \left( \frac{t-R}{\alpha} \right) \right]
\]

(1.51)

\[-\frac{1}{R\beta^2} \left( \gamma_i\gamma_j - \delta_{ij} \right) f \left( t - \frac{R}{\beta} \right),
\]

where

\[
\gamma_i = \frac{x_i - \xi_i}{R}.
\]

Similarly following similar procedure as above the Green’s function in a half space can be obtained on substituting the force system in Eq. (1.41) and the result agrees with Johnson (1974).
1.9 Principle of Fracture Mechanics

All things on Earth, be it a building, dam, aircraft or any material, perish with time due to fracture in the material. Fracture initiates near an incipient crack which is initially in crystalline form, growing from atomistic scale to macro crack. Finally, macro cracks grow in size leading to the final destruction of the material or may stop growing due to frictional resistance that comes into play. In the ruptured region, which we call a crack, the traction is suddenly relieved and drops to the difference between the initial stress and the frictional stress. The region ahead of the crack tip is called the process zone and is associated with usually irreversible process of microscopically quite complex breaking of bond forces, which can not be described in terms of the classical continuum mechanics. In metals, and for the majority of brittle materials, possible inelastic processes including plastic deformations are restricted to a small region of the process zone and can be neglected from macroscopic point of view. Then the cracked body can be assumed as linear elastic on the whole body and the law of linear fracture mechanics can be applied.

Griffith (1920) first proposed the energy criterion governing the onset of crack propagation. Fracture of a material component, literally speaking, is the breaking of the material in two or more parts. Within the ambit of linear fracture mechanics, fracture is thus the creation of a new surface which leads to a release of the elastic energy from the body while the strain energy inside the process zone decreases. The energy criterion in the general form is

\[ \delta (E + K) + \delta \Gamma = \delta P + \delta Q \]

where \( \Gamma \) is the fracture surface energy, \( K \), the kinetic energy, \( Q \), the non-mechanical energy in the form of heat, \( E \) is the internal energy and \( P \) is the work done by the external load.

In the quasi-static case \( \delta K \) and \( \delta Q \) can be neglected. According to Griffith, the fracture surface energy \( d\Gamma \) is related as \( d\Gamma = 2\gamma dA \). Then the energy criterion is

\[ \frac{d\Pi}{dA} + \frac{d\Gamma}{dA} = 0, \]

where

\[ \Pi = \iint (W - \tau_{ij}n_iu_j)dA \quad (1.52) \]

Then, noting that \( G \), the total energy release is

\[ G = -\frac{d\Pi}{dA} \frac{d\Gamma}{dA} = 2\gamma = G_c \quad (1.53) \]

\( G_c \) is the critical energy. Thus, in general, the fracture criterion can be stated as \( G \leq G_c, G_c = \text{constant.} \)

The equality corresponds to a state of neutral equilibrium.
1.9.1 Irwin’s Fracture Criterion

Major parts of this book are concerned with the indentation and crack problem. We shall find that Hertzian contact generates conical fracture at the contact edge. This is mathematically inherent in the solution of the singular integral equations which requires crack edge conditions. The mathematical singularity results in the singular stress as $\sigma_{ij} \sim K_i (2\pi r)^{-1/2}$, where $K_i$ is called the stress intensity factor or simply SIF. $K_i$ plays an important role in deriving the fracture criterion. Irwin conjectured that this singular stresses at the crack tip supply the necessary energy for the creation of the new surface as the crack propagates. The energy release rate is

$$d\Pi \sim \frac{\kappa + 1}{8\mu} K_i^2 da$$

For general loading when all the three modes are present, the crack energy is

$$G = \frac{d\Pi}{da} = \frac{1}{2\pi} \left[ (1-\nu) K_i^2 + K_{II}^2 + K_{III}^2 \right]$$

Thus, the Griffith’s fracture criterion for normal loading $G = G_c$ can be written as

$$K_i = K_{IC}.$$  \hspace{1cm} (1.55)

Eq. (1.55) is known as Irwin’s fracture criterion. $K_{IC}, G_{IC}$ are known as the fracture toughness and the crack resistance force, respectively. A crack will begin to move if the stress intensity factor increases to $K_{IC}$ for mode I plain strain deformation. Irwin’s fracture criterion is widely used in engineering application and the fracture toughness $K_{IC}$ are available for many geometrical configurations in standard engineering handbooks. $K_{IC}$ is the value of $K_i$ at the time of crack extension. It defines the onset of crack extension, but does not indicate whether the material will fracture. Despite the stress at the crack tip being infinite, the Griffith’s energy balance criterion must be satisfied for the crack to extend under applied stress. However Griffith’s energy criterion is preferred in interface cracks, particularly in composite material or laminates.

1.9.2 Other Fracture Criteria

Many other fracture criteria such as $J$ integral, strain energy density theory have been suggested Sih (1972), Rice (1968) using the conservation integrals proposed

$$J = \int_{r} \left( Wdy - T \cdot \frac{\partial u}{\partial x} \right) dx$$

(1.56)

As the fracture criterion and $T$ is the traction vector in the outward normal along $\Gamma$ and $\Gamma$ is a curve surrounding the crack tip.

For quasi-static linear fracture $J = G$ \hspace{1cm} (1.57)

Thus $J = J_c$ is the alternate fracture criterion and can be readily used for plastic and inelastic deformation.
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