Mathematical problems of classical nonlinear electromagnetic theory
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This book is dedicated
to the
memory of my Brother
DONALD NORMAN MARTIN
née Bloom
(1933–1989)

and

to the
memory of my Mother
JEANETTE BLOOM
(1909–1991)
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Preface

This book is a survey of a collection of problems arising in classical nonlinear electromagnetic theory with which the author has been involved for more than twelve years. The subject matter treated here has its origins in one of three basic subareas of nonlinear electromagnetic theory, namely, the propagation of electromagnetic waves in nonlinear dielectric materials, the evolution of the charge and current distributions in a nonlinear transmission line, and nonlocal problems in electromagnetic theory which depend, for their formulation, on the Biot–Savart Law. During the past several decades, it would seem that only two surveys have appeared in book form which touch on the broad range of subject matter covered here: the volume by I. G. Kataev on electromagnetic shock waves, which was published in 1966, and the monograph by A. C. Scott, whose focus was on wave propagation in transmission lines and which appeared in 1980; almost all of the results presented in the current volume have appeared since the publication of the two aforementioned surveys. It is hoped that this monograph will not only serve to introduce the reader to a beautiful (and somewhat neglected) area of applied mathematics, but that it may have the much-desired effect of stimulating further research on the many problems in this general area which are still unresolved, especially those which must be formulated in several space dimensions.

For the mathematician who has had little (or no) exposure to the subject matter of classical electromagnetic theory, as it is presented in most standard beginning graduate courses in American physics departments, we present, in Chapter 1, a brief survey of those parts of the subject which are deemed most useful to understanding what follows in the remainder of the book; the treatment in this first chapter, while concise, is strongly dependent on the presentation in several standard graduate level texts on electromagnetic theory.

In Chapter 2 we introduce the subject of wave propagation in nonlinear dielectrics, focusing on the problem of singularity formation in the wave-dielectric interaction problem, with much of the emphasis on problems which are dispersionless and can be formulated in one space dimension; the chapter begins with a survey of basic shock wave theory. Chapter 3 continues the discussion of the wave-dielectric interaction
problem and is concerned with the derivation of growth estimates for solutions of the relevant initial-boundary value problems, as well as with the general issues of existence of smooth solutions for these problems, globally in time, and the asymptotic behavior of such solutions for large time.

Chapter 4 initiates our study of wave propagation in a distributed parameter nonlinear transmission line and covers both the issues of shock formation for "large" data, and the existence of smooth solutions for "small" data; in the discussion of both of these general problem areas we deal with a dispersionless line. In Chapter 5 we return to the transmission line problem to deal with the issue of existence of globally defined weak solutions in the presence of "large" initial data; the treatment in this chapter necessitates the introduction of the Young measure and the general concept of compensated compactness which have been instrumental, in recent years, in treating a variety of important nonlinear problems that arise in physical applications.

Chapter 6 offers a discussion of some nonlocal problems of electromagnetic theory with an emphasis on equilibrium problems that involve a self-interaction term arising from an application of the Biot–Savart Law; a particular focus is on problems for nonlinearly elastic, self-interacting current bearing wires in an ambient magnetic field.

This book was begun shortly after the tragic death of my brother during the summer of 1989 and was finished after the passing of my mother in the fall of 1991; it is dedicated, with much affection, to both of them. At those times in an individual's life when one must endure an emotional rollercoaster ride, it may be deemed a privilege to be a working scientist and to be involved with subject matter which appears to transcend, in a way that none of us yet seems to understand completely, our own mundane existence as human beings.

While the work of many colleagues has been referenced in this volume, several of them deserve to be singled out for special mention: Professor Alan Jeffrey, whose earlier work on some of the problems discussed in this book was instrumental, many years ago, in getting the author interested in these problems in the first place; Professors Constantine Dafermos and Marshall Slemrod, both friends of many years, whose mathematical work has inspired considerable portions of the material found in this book; Professor Stuart Antman, whose collaboration led to much of the work on nonlocal problems presented in the last chapter of this volume and, especially,
my colleagues Professors Hamid Bellout and Jindřich Nečas, who have been friends as well as collaborators on many of the problems which are treated in this monograph. A special thanks goes to Mari–Anne Hartig, whose expert preparation of a long manuscript is very much appreciated by the author. Much of the work reported here, which is related to my own research in this area, was supported during the years 1984–1990 by a series of grants from the Applied Mathematics Program at NSF, for which the author is sincerely grateful. Finally, it is a pleasure to acknowledge the support of my family, my wife Leah, and my sons Daniel and Amir, whose constant encouragement during difficult times has made this book possible.

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Chapter 1

ELEMENTS OF CLASSICAL ELECTROMAGNETIC THEORY

1.0 Introduction

Our goal in this first chapter is to provide the reader with a concise review of those elements of classical electromagnetic theory which will be particularly useful with regard to understanding the physical content of the remaining chapters. All of the material presented in this review is standard and may be gleaned from any one of the number of excellent graduate level texts available on the subject, e.g., [134], [73], [166]; however, even the reader who is more or less familiar with the structure and content of classical electromagnetic theory may find it helpful to quickly thumb through the material presented below in this introductory chapter. Our review of classical electromagnetic theory adheres to the following order: In § 1.1, we offer a quick synopsis of the classical theory of electrostatics emphasizing the theory associated with electrostatic fields in dielectric materials; the concepts of electric fields and potentials are introduced, as well as those of polarization and electric displacement, which will be so central to the material of Chapter 2. We continue in § 1.2 with a discussion of electric currents and electromagnetic induction, introducing, in the process, the current density vector (which also figures prominently in the discussion of wave-dielectric interactions in the next chapter) and the associated Ohm’s laws (both the linear and nonlinear versions). Also introduced in § 1.2 are the concepts of resistance and magnetic induction, as well as the Biot-Savart law upon which much of the discussion in Chapter 6 concerning self-interacting current bearing wires revolves. The work in § 1.2 concludes with an analysis of the concepts of magnetic flux and electromotive force and leads us to the differential form of Faraday’s law. Ampere’s circuital law, which is yet one other consequence of the analysis in § 1.2, is generalized in § 1.3, in the manner first carried out by Maxwell, so as to take into account situations in which the charge density may change with time; this leads us, in § 1.3, to
the full set of Maxwell's equations which are used in a classical fashion to investigate wave propagation in a linear medium. Our work in § 1.3 concludes, however, with the derivation of a coupled set of damped nonlinear wave equations for the components of the electric displacement field in a rigid nonlinear dielectric satisfying a simple nonlinear Ohm's law. Finally, in § 1.4, we present a discussion of the basic concepts associated with electrical circuits (or transmission lines); this includes an analysis of the concepts of capacitor, resistor, and inductor as components of a circuit and the delineation of Kirchhoff's laws in circuits excited by both constant and slowly varying voltages. In all the work in § 1.4 (unlike the analysis to be presented in Chapter 4), it is assumed that the resistance, capacitance, and self-inductance in the transmission line are constants and we exclude the presence in the circuit (until Chapter 4) of a leakage conductance per unit length of the line. The presentation in § 1.4 concludes with a standard analysis of the steady-state and transient behavior of the current in a series RLC transmission line.

1.1 Electrostatics and Dielectric Media

Electrostatic theory is based on Coulomb's law, which for a point charge $\bar{q}$, at the origin of some chosen coordinate system, and a point charge $q$ at $r$, gives the electrostatic force on $q$ as

$$F_e = \frac{1}{4\pi\varepsilon_0 r^2} \frac{q\bar{q}}{r}$$

(1.1.1)

In (1.1.1), $r$ is the position vector from the origin to the point occupied by the charge $q$, while $\varepsilon_0 = 8.854 \times 10^{-12} \text{C}^2/\text{N}\cdot\text{m}^2$ in mks units$^1$; the constant $\varepsilon_0$ is known as the permittivity of free space. With the assumption that we are working in a Cartesian coordinate system $(x, y, z)$, $r = \|r\| = \sqrt{x^2 + y^2 + z^2}$, i.e., the usual Euclidean norm.

If we consider $q$ to be a test charge, then we may use (1.1.1) to define the electric field $E$ corresponding to the electric force $F_e$ by $F_e = qE$, so that the electrostatic field at $r$ which is due to a source charge $\bar{q}$ placed at $r = 0$ is given by

$$E(r) = \frac{1}{4\pi\varepsilon_0 r^2} \frac{\bar{q}}{r}$$

(1.1.2)

---

$^1$In Gaussian units $\varepsilon_0 = \frac{1}{4\pi}$. 


For a system of \( n \) charges \( q_i, i = 1, \ldots, n \), localized at positions \( \mathbf{r}_i \), (1.1.1) generalizes in the expected fashion so as to produce, for the force on the \( i \)-th charge,

\[
F_i = q_i \sum_{k \neq i}^{n} \frac{q_k \mathbf{r}_{ik}}{4\pi \varepsilon_0 r_{ik}^3}
\]

(1.1.3)

where \( \mathbf{r}_{ik} = \mathbf{r}_i - \mathbf{r}_k \) and \( r_{ik} = ||\mathbf{r}_{ik}|| \). In view of the fact that \( \nabla \times \left( \frac{\mathbf{r}}{r^3} \right) = 0 \), while \( \nabla \cdot \left( \frac{\mathbf{r}}{r^3} \right) = 4\pi \delta(\mathbf{r}) \), \( \delta \) being the Dirac delta function (distribution), we find that for a point charge (1.1.2) yields

\[
\nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \hat{q} \delta(\mathbf{r})
\]

(1.1.4)

For a continuous distribution of charge density \( \rho(\mathbf{r}) \) defined over a region \( \Omega \subseteq \mathbb{R}^3 \), so that the element of charge contained in a volume element \( d\mathbf{v} \) is \( dq = \rho(\mathbf{r}) d\mathbf{v} \), we have

\[
\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int_{\Omega} \frac{\mathbf{r} - \mathbf{\bar{r}}}{||\mathbf{r} - \mathbf{\bar{r}}||^3} \rho(\mathbf{\bar{r}}) d\mathbf{\bar{v}}
\]

(1.1.5)

Clearly, \( ||\mathbf{r} - \mathbf{\bar{r}}|| = \sqrt{(x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2} \) in (1.1.5). Of course, \( \rho(\mathbf{r}) = q_i \delta(\mathbf{r} - \mathbf{r}_i) \) for a point charge \( q_i \) located at \( \mathbf{r}_i \). It follows directly, as a consequence of (1.1.5), that

\[
\nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho(\mathbf{r})
\]

(1.1.6)

and these are the fundamental partial differential equations which are satisfied by all electrostatic fields.

If we integrate the second equation in (1.1.6) over \( \Omega \subseteq \mathbb{R}^3 \), and apply the divergence theorem, we obtain Gauss’s law, i.e.,

\[
\int_{\partial\Omega} \mathbf{E} \cdot \mathbf{n} dS = \frac{1}{\varepsilon_0} Q; \quad Q = \int_{\Omega} \rho(\mathbf{r}) d\mathbf{v}
\]

(1.1.7)

where \( \mathbf{n} \) is the (unit) exterior normal to \( \partial\Omega \) and \( Q \) is the net charge inside \( \Omega \). From the first equation in (1.1.6), we infer immediately the existence (for a simply connected domain \( \Omega \)) of an electrostatic potential function \( \phi(\mathbf{r}) \) satisfying \( \mathbf{E} = -\nabla \phi \): the potential \( \phi \) can be expressed either in terms of a given electric field \( \mathbf{E} \) by

\[
\phi(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}
\]

(1.1.8)

or in terms of a given charge distribution \( \rho(\mathbf{r}) \) by

\[
\phi(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \int_{\Omega} \frac{\rho(\mathbf{r})}{||\mathbf{r} - \mathbf{\bar{r}}||} d\mathbf{\bar{v}}
\]

(1.1.9)
Solutions of electrostatic problems in a domain $\Omega$ are obtained by combining the second equation in (1.1.6) with the consequence of the first equation, i.e., the existence of the potential function $\phi(\mathbf{r})$, so as to produce the Poisson equation $\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$, which is then solved subject to the specification of either $\phi$ or $\frac{\partial \phi}{\partial n}$ on $\partial \Omega$.

When we have two equal and opposite charges separated by a small distance, we say that an electric dipole exists. For a charge $-q$ located at $\mathbf{r}$ and a charge $q$ located at $\mathbf{r} + \mathbf{l}$, the electric field is readily calculated to be

$$E(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \left\{ \frac{\mathbf{r} - \mathbf{r} - \mathbf{l}}{||\mathbf{r} - \mathbf{r} - \mathbf{l}||^3} - \frac{\mathbf{r} - \mathbf{r}}{||\mathbf{r} - \mathbf{r}||^3} \right\}$$

(1.1.10)

For a dipole field, the separation $||\mathbf{l}||$ is small compared with $||\mathbf{r} - \mathbf{r}||$; expanding (1.1.10) by means of the binomial theorem, and retaining only those terms linear in $\mathbf{l}$, yields

$$E(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \left\{ \frac{3(\mathbf{r} - \mathbf{r}) \cdot \mathbf{p}}{||\mathbf{r} - \mathbf{r}||^5} \cdot (\mathbf{r} - \mathbf{r}) - \frac{\mathbf{p}}{||\mathbf{r} - \mathbf{r}||^3} \right\}$$

(1.1.11)

where $\mathbf{p} = q\mathbf{l}$ is the electric dipole moment. For a point dipole it is assumed that $||\mathbf{l}|| \to 0$, while $q \to \infty$, in such a way that $||\mathbf{p}||$ remains constant; in this case all terms in the expansion of (1.1.10) vanish in the limit, except those which are linear in $\mathbf{l}$, and (1.1.11) is exact for the point dipole. The potential corresponding to a point dipole may easily be shown to have the form

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r})}{||\mathbf{r} - \mathbf{r}||^3}$$

(1.1.12)

For a continuous distribution $\rho(\mathbf{r})$ of charge throughout a domain $\Omega \subseteq \mathbb{R}^3$, the dipole moment of the distribution is defined to be

$$\mathbf{p} = \int_{\Omega} \mathbf{r} \rho(\mathbf{r}) d\mathbf{v}$$

(1.1.13)

In this treatise considerable attention will be paid to the matter of electromagnetic wave-dielectric interactions (i.e., all of Chapters 2 and 3); as background material we now survey briefly the theory underlying electrostatic fields in dielectric media. Although an ideal dielectric material is one in which there exists no free charge, the term bound charge, when used in reference to dielectrics (as opposed to free charge for a conductor) is used to emphasize the fact that molecular charges are not free to travel significant distances or to be extracted from the dielectric; under the
action of an electric field, the entire positive charge in a dielectric is viewed as being displaced relative to the negative charge and the dielectric is said to be polarized. A polarized dielectric, while electrically neutral (on the average) produces an electric field, both interior and exterior to the dielectric, which may, in turn, modify the free charge distribution on any conducting bodies in its vicinity and lead to changes in the external electric field which act back on the dielectric. If $\Delta v$ denotes an infinitesimal volume element of a polarized dielectric, this volume element is characterized by an electric dipole moment which is given, according to (1.1.13), by

$$\Delta p = \int_{\Delta v} r \, dq$$

(1.1.14)

and gives rise to an electric dipole moment (per unit volume)

$$P = \lim_{\Delta v \to 0} \frac{\Delta p}{\Delta v}$$

(1.1.15)

When the limit in (1.1.15) exists, the point function $P$ is termed the electric polarization of the dielectric medium and the electrostatic potential at a point $r$ exterior to the dielectric, assumed to occupy a domain $\Omega \subseteq R^3$, is then given, according to (1.1.12), by

$$\phi(r) = \frac{1}{4\pi\varepsilon_0} \int_{\Omega} \frac{P(\bar{r}) \cdot (r - \bar{r}) d\bar{v}}{||r - \bar{r}||^3}$$

(1.1.16)

Inasmuch as $\nabla \left( \frac{1}{||r - \bar{r}||} \right) = \frac{r - \bar{r}}{||r - \bar{r}||^3}$, where $\nabla$ is the gradient operator with respect to the barred coordinates, (1.1.16) may be transformed into

$$\phi(r) = \frac{1}{4\pi\varepsilon_0} \int_{\partial\Omega} \frac{P \cdot n}{||r - \bar{r}||} d\bar{S}$$

$$+ \frac{1}{4\pi\varepsilon_0} \int_{\Omega} \left( -\nabla \cdot P(\bar{r}) \right) d\bar{v}$$

(1.1.17)

through use of the standard identity

$$\nabla \cdot (f P) = f \nabla \cdot P + P \cdot \nabla f$$

with $f = \frac{1}{||r - \bar{r}||}$. The quantities $\sigma_p = P \cdot n$ and $\rho_p = -\nabla \cdot P$ are, respectively, the surface and volume densities of polarization charge. Inserting these quantities in (1.1.17), we see that we may express the potential due to the dielectric medium in such a way that it arises from appropriate distributions of polarization charge, i.e.,

$$\phi(r) = \frac{1}{4\pi\varepsilon_0} \left\{ \int_{\partial\Omega} \frac{\sigma_p d\bar{S}}{||r - \bar{r}||} + \int_{\Omega} \frac{\rho_p d\bar{v}}{||r - \bar{r}||} \right\}$$

(1.1.18)
the definitions of $\sigma_p, \rho_p$ being obvious upon comparison with (1.1.17).

We now recall the statement of Gauss's law, which states that the electric flux across an arbitrary closed surface is proportional to the total charge enclosed by the surface; in applying Gauss's law to a domain $\Omega \subseteq \mathbb{R}^3$ occupied by a dielectric medium, the polarization charge, as well as the charge embedded in $\Omega$, must be included. If $\partial\Omega'$ bounds a domain $\Omega' \subset \Omega$ and contains an amount of embedded charge $Q = \sum_{i=1}^{n} q_i$, with $q_i, i = 1, \ldots, n$ existing on the surfaces $\partial\Omega_i$ of $n$ conductors, then by Gauss's law

$$\oint_{\partial\Omega'} E \cdot n \, dS = \frac{1}{\varepsilon_0} (Q + Q_p)$$

(1.1.19)

where $Q_p$ is the net polarization charge, i.e.

$$Q_p = \int_{\partial\Omega_0} P \cdot n \, dS + \int_{\Omega'} (-\nabla \cdot P) \, dv'$$

(1.1.20)

The surface integral in (1.1.20) does not contain a contribution from $\partial\Omega'$ inasmuch as there is no boundary of the dielectric there. Using the divergence theorem on the volume integral in (1.1.20), and noting that $\Omega'$ is bounded by $\partial\Omega' \cup (\bigcup_i \partial\Omega_i)$, we easily find that

$$Q_p = -\int_{\partial\Omega'} P \cdot n \, dS$$

(1.1.21)

Combining (1.1.19) with (1.1.21) then yields

$$\oint_{\partial\Omega'} (\varepsilon_0 E + P) \cdot n \, dS = Q$$

(1.1.22)

Equation (1.1.22) naturally yields the definition of the field vector

$$D = \varepsilon_0 E + P$$

(1.1.23)

which is termed the electric displacement, in terms of which (1.1.22) becomes (we drop the prime)

$$\oint_{\partial\Omega} D \cdot n \, dS = Q$$

(1.1.24)

and (1.1.24) applies to any region $\Omega$ of $\mathbb{R}^3$ bounded by a closed surface $\partial\Omega$; applying (1.1.24) to an infinitesimal domain $\Omega \subseteq \mathbb{R}^3$, with continuous distributed charge density $\rho$, dividing both sides of (1.1.24) by $\text{vol}(\Omega)$, and extracting the limit as $\text{vol}(\Omega) \to 0$, we obtain the differential form of Gauss's law, namely,

$$\nabla \cdot D = \rho$$

(1.1.25)
The degree of polarization in a dielectric depends not only on the impressed electric field, but also on the molecular properties of the material; macroscopically this behavior is specified by an experimentally determined relationship of the form $P = P(E)$.\footnote{Constitutive relations in which $P$ depends both on the electric and magnetic fields may also occur.} Inasmuch as it is usually the case that $E = 0$ implies that $P = 0$, and that the dielectric is isotropic, so that the polarization vector points in the same direction as the impressed electric field, a common form of the $P$-$E$ relationship is $P = \chi(||E||)E$, where $\chi(\cdot)$ is a scalar-valued function called the electric susceptibility; in this case it is clear that by virtue of (1.1.23) we have

$$D = \varepsilon(||E||)E$$

where the permittivity $\varepsilon(\cdot)$ is defined to be

$$\varepsilon(||E||) = \varepsilon_0 + \chi(||E||)$$

For many dielectric materials, $\chi(\cdot)$, and hence $\varepsilon(\cdot)$, are independent of the field strength $||E||$, except for intense electric fields of the type common in laser beams; for such linear dielectrics, $\chi$ and $\varepsilon$ are constants characteristic of the dielectric medium.

If we consider two dielectrics which are in contact along a common boundary $S$ (a vacuum may be considered to be a dielectric with permittivity $\varepsilon_0$) which carries a surface density of external charge $\sigma$, then an application of Gauss's law to a pillbox-shaped surface $S'$, which intersects the interface $S$, yields the relation

$$(D_2 - D_1) \cdot n = \sigma$$

where $D_i$ is the value of the displacement field in the $i$-th dielectric at a point $x \in S$ and $n$ is the exterior unit normal to $S$ at $x$; when $\sigma = 0$ on $S$, (1.1.28) states that the normal component of $D$ is continuous across the interface between the two media.

In an analogous fashion, as $E = -\nabla \phi$, the line integral $\oint E \cdot dl$ around any closed path vanishes, and the application of this result to an arbitrary infinitesimal rectangular path which intersects the interface $S$ yields the second fundamental boundary condition for dielectric media, namely

$$(E_2)_t = (E_1)_t$$

(1.1.29)
where the subscript $t$ denotes the tangential component of the indicated vector field. The boundary conditions (1.1.28), (1.1.29), in conjunction with (1.1.25), and the constitutive relation (1.1.26), yield a well-posed boundary value problem for the electrostatic field in any dielectric medium (or vacuum); if the first medium is both linear and isotropic, so that $\mathbf{D} = \epsilon \mathbf{E}$, as well as $\mathbf{E} = -\nabla \phi$, then (1.1.25) yields the Poisson equation

$$\nabla^2 \phi = -\frac{1}{\epsilon} \rho$$

(1.1.30)

for the electrostatic potential in the dielectric.

Before concluding this section on electrostatics and dielectric media, some remarks are in order about the electrostatic energy of a charge distribution and the corresponding concept of energy density in an electrostatic field. It is easily demonstrated that the electrostatic energy of an arbitrary charge distribution, which possesses volume density $\rho$ and surface density $\sigma$, is given by

$$E = \frac{1}{2} \int_{\Omega} \rho(\mathbf{r}) \phi(\mathbf{r}) \, dv + \frac{1}{2} \int_{\partial \Omega} \sigma(\mathbf{r}) \phi(\mathbf{r}) \, dS$$

(1.1.31)

provided that all dielectrics present are linear, the domain of integration $\Omega$ is large enough to include all of the charge density present, and the electrostatic potential $\phi$ is that due to the charge densities $\rho$ and $\sigma$; for a domain $\Omega \subseteq \mathbb{R}^3$ filled with a single, linear dielectric medium, of dielectric permittivity $\epsilon$, the potential $\phi$ is given by the expression

$$\phi(\mathbf{r}) = \frac{1}{4\pi \epsilon} \int_{\Omega} \frac{\rho(\mathbf{r})}{||\mathbf{r} - \mathbf{\tilde{r}}||} \, d\mathbf{\tilde{r}} + \frac{1}{4\pi \epsilon} \int_{\partial \Omega} \frac{\sigma(\mathbf{\tilde{r}})}{||\mathbf{r} - \mathbf{\tilde{r}}||} \, d\mathbf{\tilde{S}}$$

(1.1.32)

The corresponding situation for point charges follows as a special case of (1.1.31), (1.1.32) if we set

$$\begin{cases}
\rho(\mathbf{r}) = \sum_{n=1}^{m} q_n \delta(\mathbf{r} - \mathbf{r}_n) \\
\rho(\mathbf{\tilde{r}}) = \sum_{k=1}^{m} q_k \delta(\mathbf{\tilde{r}} - \mathbf{\tilde{r}}_k)
\end{cases}$$

(1.1.33)

in which case

$$E = \frac{1}{2} \sum_{j=1}^{m} q_j \phi_j$$

(1.1.34)

with

$$\phi_j = \sum_{k=1 \atop k \neq j}^{m} \frac{q_k}{4\pi \epsilon_0 r_{jk}}$$

(1.1.35)
In a dielectric medium, we have, by virtue of (1.1.25) and (1.1.28), \( \rho = \nabla \cdot D \), 
\( \sigma = D \cdot n \), provided \( \Omega \) is constructed so that all surface densities of charge \( \sigma \) reside on conductor surfaces; then (1.1.31) becomes

\[
E = \frac{1}{2} \int_{\Omega} \nabla \cdot D \, dv + \frac{1}{2} \int_{\partial \Omega'} \phi D \cdot n \, dS' 
\]  
(1.1.36)

In (1.1.36), \( \Omega \) is the domain where \( \nabla \cdot D \neq 0 \), while the surface integral is over the surface of the conductors. Employing the standard identity for \( \nabla \cdot \phi D \), and using the divergence theorem, as well as the fact that \( E = -\nabla \phi \), the expression (1.1.36) is easily seen to reduce to

\[
E = \frac{1}{2} \int_{\Omega} D \cdot E \, dv 
\]  
(1.1.37)

**Remark.** If we refer to Figure 1.1 and set \( \partial \Omega' = \bigcup_i \partial \Omega_i \), then we note that (1.1.36) first is reduced to

\[
E = \frac{1}{2} \int_{\partial \Omega_0 \cup \partial \Omega'} \phi D \cdot n \, dS + \frac{1}{2} \int_{\Omega} D \cdot E \, dv 
\]  
+ \frac{1}{2} \int_{\partial \Omega'} \phi D \cdot n \, dS' 
\]  
(1.1.38)
The two surface integrals over \( \partial \Omega' \) mutually cancel each other; as \( \phi \sim r^{-1}, \|D\| \sim r^{-2} \), for large \( r \), while \( \partial \Omega_0 \sim r^2 \) as we extend \( \partial \Omega_0 \) out to infinity, so

\[
\int_{\partial \Omega_0} \phi \mathbf{D} \cdot \mathbf{n} \, dS \to 0.
\]

From (1.1.37) we infer that the energy density \( u_E \) in an electrostatic field is given by the expression

\[
u_E = \frac{1}{2} \mathbf{D} \cdot \mathbf{E}
\]  

(1.1.39)

### 1.2 Electric Currents and Electromagnetic Induction

We now consider charges in uniform motion and thus will be dealing with conductors of electricity, which are materials in which the charge carriers are free to move under steady electric fields; this definition of conductor includes not only standard conductors such as metallic substances, but also imperfect dielectric media, and so the charge carriers may be either electrons or positive or negative ions.

The basic definition, of course, is that current is moving charge and a current \( I \) is then defined as the rate at which charge is transported through a given surface, i.e., \( I = \frac{dQ}{dt} \); the unit of current in the mks system is the ampere, where 1 (ampere) \( = 1 \) (coulomb/second). Consider a conducting medium which possesses \( N \) charge carriers, each of charge \( q \), per unit volume, with each carrier having the same drift velocity \( v \); the net charge \( \delta Q \) which crosses an element of surface in the conductor of area \( dS \), in time \( \delta t \), is easily seen to be

\[
\delta Q = qNv \cdot \mathbf{n} \delta t \, dS
\]

(1.2.1)

where \( \mathbf{n} \) is the external unit normal to the element of surface in the conductor. From (1.2.1) it is immediate that

\[
dI \equiv \frac{\delta Q}{\delta t} = Nqv \cdot \mathbf{n} \, dS
\]

(1.2.2)

and if \( M \) different charge carriers are present, each carrying a charge \( q_i \), and numbering \( N_i, i = 1, \ldots, m \), then (1.2.1) clearly generalizes to

\[
dI = \left\{ \sum_{i=1}^{M} N_i q_i v_i \right\} \cdot \mathbf{n} \, dS
\]

(1.2.3)
The quantity in the brackets in (1.2.3) has the dimensions of current per unit area and is called the current density \( J \), i.e.,

\[
J = \sum_i N_i q_i v_i \tag{1.2.4}
\]

so that (1.2.3) may be expressed as \( dI = J \cdot n \, dS \) and the current through an arbitrary surface \( \partial \Omega \), bounding a domain \( \Omega \subseteq R^3 \) in the conductor, is then given by

\[
I = \int_{\partial \Omega} J \cdot n \, dS \tag{1.2.5}
\]

Inasmuch as the current entering \( \Omega \) through \( \partial \Omega \) is the negative of the quantity in (1.2.5), we have, by virtue of the divergence theorem,

\[
I = \int_{\Omega} \nabla \cdot J \, dv \tag{1.2.6}
\]

But, for a fixed, time-independent domain \( \Omega \subseteq R^3 \), it is also true that

\[
I = \frac{dQ}{dt} = \frac{d}{dt} \int_{\Omega} \rho(x, t) \, dv = \int_{\Omega} \frac{\partial \rho}{\partial t} \, dv \tag{1.2.7}
\]

where we now denote the position of points in \( \Omega \) by \( x = (x_1, x_2, x_3) \); combining (1.2.6), (1.2.7), assuming all indicated quantities are, at least, continuous functions of \( x \), and noting that the resulting integral relation must hold for each arbitrary bounded domain \( \Omega \) in the conductor, we are led to the equation of continuity, namely,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \tag{1.2.8}
\]

For many metallic substances at constant temperature, it has been noted experimentally that \( J \) is linearly proportional to \( E \) and the mathematical statement of this fact, i.e., \( J = \sigma E \), is termed Ohm's law; the constant of proportionality \( \sigma \) (not to be confused with a measure of surface charge density) is called the conductivity of the material. In many instances, the conductivity may be demonstrated to be field dependent, e.g., conduction of electricity takes place in an imperfect dielectric, in which case the standard Ohm's law must be replaced by a constitutive relation of the form

\[
J = \sigma(E)E \tag{1.2.9}
\]

and, indeed, nonlinear Ohm's laws of this specific type will figure prominently in our considerations of wave-dielectric interactions in Chapter 2.
If we consider a homogeneous wire of uniform cross-section, which conducts electricity according to the linear Ohm’s law \( J = \sigma E \), and has its ends maintained at a constant potential difference \( \delta \phi \), then an electric field will exist in the wire which satisfies the relation

\[
\delta \phi = \int E \cdot dl
\]  

(1.2.10)

and the electric field \( E \) is longitudinal with no component at a right angle to the axis of the wire; by virtue of the geometry of the situation described above, and the homogeneity of the wire, it is clear that (1.2.10) implies that \( \delta \phi = El \), where \( l \) is the length of the wire and \( E = \|E\| \). However, inasmuch as \( J = \sigma E \), with \( E \) longitudinal, and the cross-sectional area of the wire is constant, say \( A \), the current through any cross-section of the wire is given by

\[
I = \int_A J \cdot n \, dS = JA
\]  

(1.2.11)

From (1.2.11), the linear Ohm’s law, and the relation \( \delta \phi = El \), we easily obtain a linear relationship

\[
I = \frac{\sigma A}{l} \delta \phi = \frac{1}{R} \delta \phi
\]  

(1.2.12)

which serves to define the resistance \( R \) of the wire. It is not difficult to infer that the linear Ohm’s law \( J = \sigma E \) implies the relation (1.2.12) independently of the shape of the conductor.

For a homogeneous conducting medium in a steady-state conduction mode, which obeys a linear Ohm’s law, \( \frac{\partial \rho}{\partial t} = 0 \), and the equation of continuity (1.2.8) reduces to the statement that \( \nabla \cdot \sigma E = 0 \); with \( \sigma \) constant in a homogeneous medium, we infer that \( \nabla \cdot E = 0 \) and if the field is static, so that \( \nabla \times E = 0 \), we again infer the existence of a scalar potential \( \phi \) satisfying Laplace’s equation. The steady state conduction problem for any static system of homogeneous conductors, which conforms to a linear Ohm’s law may, therefore, be solved in the same manner as electrostatic problems; to solve a boundary value problem for a conducting medium we must, of course, specify either \( \phi \) or \( J \) at each point on the bounding surface of the medium.

Suppose we consider an isotropic homogeneous medium possessing conductivity \( \sigma \) and dielectric permittivity \( \epsilon \), i.e., for this medium, both \( J = \sigma E \) and \( D = \epsilon E \) apply. Suppose further that, at time \( t = 0 \), the medium is characterized by a prescribed
volume density of charge $\rho_0(x, y, z)$; from the equation of continuity (1.2.8) and Ohm's law we have

$$\frac{\partial \rho}{\partial t} + \sigma \nabla \cdot E = 0 \quad (1.2.13)$$

However, in view of (1.1.25) and the relation $D = \varepsilon E$, we see that $\nabla \cdot E = \frac{\rho}{\varepsilon}$ so that (1.2.13) assumes the form

$$\frac{\partial \rho}{\partial t} + \left( \frac{\sigma}{\varepsilon} \right) \rho = 0 \quad (1.2.14)$$

and integration of (1.2.14) subject to the initial condition $\rho(x, y, z, 0) = \rho_0(x, y, z)$ yields the elementary solution

$$\rho(x, y, z, t) = \rho_0(x, y, z) \exp(-\sigma t/\varepsilon) \quad (1.2.15)$$

which shows that the conducting medium, if isolated from applied electric fields, will tend to an equilibrium state exponentially fast as $t \to \infty$. The quantity $\varepsilon/\sigma$ in (1.2.15) obviously has the dimensions of time, and is termed the relaxation time of the medium.

We now turn our attention to a brief survey of those basic considerations related to a study of the magnetic field associated with steady currents in a conducting medium. We begin by recalling that the Coulomb force on a charge $q$ located at position $r$, due to a charge $\bar{q}$ at the origin of our chosen coordinate system, is given by (1.1.1), provided that both charges are at rest. When $q$ and $\bar{q}$ are, instead, moving uniformly with velocities $\mathbf{v}$ and $\bar{\mathbf{v}}$, respectively, $\bar{q}$ exerts an additional magnetic force on $q$ which is observed, experimentally, to have the form

$$\mathbf{F}_m = \left( \frac{\mu_0}{4\pi} \right) \frac{q\bar{q}}{r^2} \mathbf{v} \times \left( \frac{\bar{\mathbf{v}} \times \mathbf{r}}{r} \right) \quad (1.2.16)$$

The constant $\frac{\mu_0}{4\pi}$ in (1.2.16) plays the same role here that the constant $1/4\pi \varepsilon_0$ did in the force law (1.1.1) and $\mu_0$ is called the magnetic permeability of free space. If we now define the magnetic induction vector $\mathbf{B}$ by

$$\mathbf{B} = \left( \frac{\mu_0}{4\pi} \right) \frac{\bar{q}}{r^2} \bar{\mathbf{v}} \times \frac{\mathbf{r}}{r} \quad (1.2.17)$$

then the force law (1.2.16) assumes the form

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B} \quad (1.2.18)$$
and when both electric and magnetic fields are present the net force on a moving charge \( q \) is given by the Lorentz force

\[
F = q(E + v \times B)
\]  

(1.2.19)

We note that it is a direct consequence of (1.2.18) that, for any field \( B \), \( v \cdot F_m = 0 \) so that a magnetic force never does work on a charged particle. By multiplying both sides of (1.2.16) by \( \epsilon_0/\epsilon_0 \) we see, after comparison with the force law (1.1.1) that \( \epsilon_0\mu_0 \) has the dimensions of the square of an inverse velocity, say, \( \epsilon_0\mu_0 = c^{-2} \), with \( c \) having the dimensions of velocity; the force law (1.2.16) may then be expressed as

\[
F_m = \left( \frac{1}{4\pi\epsilon_0} \right) \frac{q\ddot{\mathbf{v}}}{r^2} \times \left( \frac{\ddot{\mathbf{v}}}{c} \times \frac{\mathbf{r}}{r} \right)
\]  

(1.2.20)

We note, in passing, that the experimentally determined values of \( \mu_0 \) and \( \epsilon_0 \) yield as a consequence that \( c = (\epsilon_0\mu_0)^{-1/2} \) is in remarkable agreement with the experimentally measured velocity of light; this agreement is, in fact, well-known to be a direct consequence of Maxwell's equations and the assumption that light is a propagating electromagnetic wave (as we will observe in the next section). We also note here that it is a direct consequence of (1.1.1) and (1.2.20) that

\[
\frac{\|F_m\|}{\|F_e\|} \leq \frac{\|\mathbf{v}\|\|\ddot{\mathbf{v}}\|}{c^2}
\]  

(1.2.21)

so that if the particle velocities are small, in comparison to the velocity of light, then the magnetic force exerted is far smaller than the electric force.

From the magnetic force relation (1.2.18) one may compute an expression for the force exerted on an element \( dl \) of a current-carrying conductor. Assuming that \( dl \) is always in the direction of the current \( I \) in the conductor, so that \( dl \) is parallel to the velocity \( \mathbf{v} \) of the charge carriers, we have, with \( K \) charge carriers per unit volume in the conductor, a force exerted on the element \( dl \) of the form

\[
dF = qKA\|d\mathbf{l}\|\mathbf{v} \times \mathbf{B}
\]  

(1.2.22)

with \( A \) being the constant cross-sectional area of the conductor and \( q \) the charge on each charge carrier. Because \( \mathbf{v} \) and the element \( dl \) are parallel, we may rewrite (1.2.22) in the form

\[
dF = qKA\|\mathbf{v}\|d\mathbf{l} \times \mathbf{B}
\]  

(1.2.23)
and this expression is unchanged if more than one type of charge carrier is involved. If we examine (1.2.23), and note that the quantity \( qKA\|v\| \) is the current \( I \) associated with one type of charge carrier, then this equation for the force on an infinitesimal element of a charge-carrying conductor may be expressed as

\[
d\mathbf{F} = I\, dl \times \mathbf{B}
\]

(1.2.24)

If \( C \) is a closed circuit in \( \mathbb{R}^3 \), and we integrate both sides of (1.2.24) over \( C \), we obtain the force exerted on the current carrying circuit in the form

\[
\mathbf{F} = I \oint_C dl \times \mathbf{B}
\]

(1.2.25)

from which it is clear that \( \mathbf{F} = 0 \) when \( \mathbf{B} \) is a uniform field (i.e., when \( \mathbf{B} \) is independent of position). The infinitesimal torque on an element \( dl \) of the conductor may be expressed as \( d\mathbf{r} = r \times d\mathbf{F} \) so that, by virtue of (1.2.24), the torque \( \mathbf{r} \) on a closed circuit \( C \) is given by

\[
\mathbf{r} = I \oint_C r \times (dl \times \mathbf{B})
\]

(1.2.26)

For a uniform field \( \mathbf{B} \) an elementary analysis shows that (1.2.26) may be reduced to the relation \( \mathbf{r} = I\mathbf{A}^* \times \mathbf{B} \) with \( \mathbf{A}^* \) being the vector whose components are the areas of the projections of \( C \) on the \( yz \), \( zx \), and \( xy \) planes, respectively. It is customary to write \( \mathbf{m} = I\mathbf{A}^* \) and to refer to \( \mathbf{m} \) as the magnetic moment of the circuit; inasmuch as

\[
\mathbf{A}^* = \frac{1}{2} \oint_C r \times dl
\]

(1.2.27)

we may express the magnetic moment as

\[
\mathbf{m} = \frac{1}{2} I \oint_C r \times dl
\]

(1.2.28)

The discovery that currents produce magnetic effects was made by Oersted in 1820 and was followed, very closely, by Ampere's results on the magnetic interaction of two current-carrying circuits. For a circuit \( C_1 \) (whose points are located by the position vector \( r_1 \), which has line element \( dl_1 \), and carries a current \( I_1 \)) and a second circuit \( C_2 \) (whose points are located by the position vector \( r_2 \), which has line element \( dl_2 \), and carries a current \( I_2 \)), Ampere's experiments imply that the force \( \mathbf{F}_2 \) exerted on \( C_2 \) due to the influence of \( C_1 \) has the form

\[
\mathbf{F}_2 = \left( \frac{\mu_0}{4\pi} \right) I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{dl_2 \times [dl_1 \times (r_2 - r_1)]}{\|r_2 - r_1\|^3}
\]

(1.2.29)
and, although not obvious from the form of the relation (1.2.29), it may be shown that \( F_2 = -F_1 \), where \( F_1 \) is the force exerted on the circuit \( C_1 \) because of the influence of the magnetic field generated by the current flowing in \( C_2 \). In view of (1.2.25) we easily find that

\[
B(r_2) = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{dI_1 \times (r_2 - r_1)}{||r_2 - r_1||^3}
\]  

(1.2.30)

and this is the Biot-Savart law which will form the basis for the work on nonlocal electromagnetic problems that is described in Chapter 6. For a continuous distribution of current, described by the current density \( J(r) \), we observe that the analog of (1.2.30) applies, namely,

\[
B(r_2) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{J(r_1) \times (r_2 - r_1)}{||r_2 - r_1||^3} \, dv
\]  

(1.2.31)

and it is, in fact, an experimental deduction that all magnetic induction fields can be described in terms of a current distribution, i.e., that \( B(r) \) has the form (1.2.31) for some \( J(r) \). This observation is the basis for the conclusion that isolated magnetic poles do not exist, for if we use the vector identity

\[
\nabla \cdot (R \times S) = -R \cdot (\nabla \times S) + S \cdot (\nabla \times R),
\]  

(1.2.32)

in taking the divergence on both sides of (1.2.31), we find that

\[
\nabla_2 \cdot B(r_2) = -\frac{\mu_0}{4\pi} \int_{\Omega} \frac{J(r_1) \cdot \nabla_2 \times \left[ \frac{r_2 - r_1}{||r_2 - r_1||^3} \right]}{||r_2 - r_1||^3} \, dv
\]  

(1.2.33)

But

\[
\frac{r_2 - r_1}{||r_2 - r_1||^3} = \nabla_2 \left( \frac{-1}{||r_2 - r_1||} \right)
\]  

(1.2.34)

and as \( \nabla \times (\nabla f) = 0 \) for any scalar function \( f \), it is immediate from (1.2.33) that

\[
\nabla_2 \cdot B(r_2) = 0
\]  

(1.2.35)

Equation (1.2.35) is the first of the two basic differential laws of magnetostatics, the second of which is Ampere's law. Suppose that the magnetic induction field which is described by (1.2.31) is generated by a steady current, i.e., that \( J \) satisfies \( \nabla \cdot J = 0 \). If we compute the curl of the expression in (1.2.31), we have that

\[
\nabla_2 \times B(r_2) = \frac{\mu_0}{4\pi} \int_{\Omega} \left\{ J(r_1) \left( \nabla_2 \cdot \left[ \frac{r_2 - r_1}{||r_2 - r_1||^3} \right] \right) - J(r_1) \cdot \nabla_2 \left[ \frac{r_2 - r_1}{||r_2 - r_1||^3} \right] \right\} dv
\]  

(1.2.36)
Changing from $\nabla_2$ to $\nabla_1$ in (1.2.36), and using the symmetry between $r_2$ and $r_1$, we compute that

$$\nabla_2 \times B(r_2) = \frac{\mu_0}{4\pi} \int_{\Omega} \left\{ 4\pi J(r_1) \delta(r_2 - r_1) \right\} dv$$

$$-J(r_1) \cdot \nabla_2 \left[ \frac{r_2 - r_1}{\|r_2 - r_1\|^3} \right]$$

where $\delta(r)$ is the Dirac delta function. An integration by parts applied to the second term in the integral in (1.2.37), when combined with the fact that $\nabla_1 \cdot J(r_1) = 0$, and followed by an application of the divergence theorem, shows that

$$\int_{\Omega} J(r_1) \cdot \nabla_1 \left[ \frac{r_2 - r_1}{\|r_2 - r_1\|^3} \right] dv = 0$$

provided $\partial\Omega$ contains the support of $J(r)$; from what remains of the relation (1.2.37) it is now immediate that

$$\nabla_2 \times B(r_2) = \mu_0 J(r_2)$$

which is the differential form of Ampere’s Law. By applying Stokes’ theorem to (1.2.39), we obtain as additional information that

$$\oint_C B \cdot dl = \int_S (\nabla \times B) \cdot n \, da \equiv \mu_0 \int_S J \cdot n \, da$$

where $C$ is any closed circuit in $\mathbb{R}^3$ that bounds the surface $S$ having an area element $da$; thus the line integral of $B$ around any closed circuit in $\mathbb{R}^3$ is equal to $\mu_0$ times the total current passing through the surface $S$ bounded by the circuit.

From the fact that $B$ is divergence free, it follows that there exists a vector field $A$, the magnetic vector potential, such that

$$B = \nabla \times A$$

In view of Ampere’s law, (1.2.39), $A$ must also satisfy $\nabla \times \nabla \times A = \mu_0 J$; if we make use of the vector identity $\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$ and specify, without loss of generality, that $\nabla \cdot A = 0$, we obtain for $A$ the relation

$$\nabla^2 A = -\mu_0 J$$

which, it may be shown, easily leads to the representation

$$A(r) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{J(\overline{r})}{\|r - \overline{r}\|} d\overline{r}$$
18 ELEMENTS OF CLASSICAL ELECTROMAGNETIC THEORY

An important quantity in magnetostatics and, especially, in discussions of electromagnetic induction, has yet to be introduced; this is the magnetic flux $\Phi$ defined by

$$\Phi = \int_S B \cdot n \, da$$  \hspace{1cm} (1.2.44)

An application of the divergence theorem to (1.2.44), when coupled with (1.2.35), shows that $\Phi = 0$ for any closed surface $S$; a further consequence of this result is, of course, that the flux through a circuit $C$ in $\mathbb{R}^3$ is independent of the particular surface (bounded by $C$) which is used to compute the flux $\Phi$.

To this point, the basic physical laws, in differential form, for describing the magnetic effects of currents, are expressed by the equations (1.2.35) and (1.2.39); these equations, however, must be modified when the magnetic field includes a contribution from a magnetized material. If a current-carrying circuit consists of a closed loop of wire, then the magnetic field in the vacuum region which surrounds the wire can be computed using (1.2.30); but, if the region which surrounds the wire is filled with a material medium, then the magnetic induction is observed experimentally to be altered by the presence of the medium and this alteration is due to nothing more than the atomic currents which result from the motion of the electrons in the atoms comprising the material medium. Each atomic current, in turn, may be described as a magnetic dipole and, if the magnetic moment of the $j$-th atom in the medium is denoted by $m_j$, then the macroscopic vector quantity $M$, the magnetization, may be defined by the same procedure which led to the definition of the polarization vector $P$, i.e. (1.1.15); we vectorially sum all the atomic dipole moments in an infinitesimal volume element $\Delta v$, divide by $\Delta v$, and extract the limit as $\Delta v \to 0$, namely,

$$M = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_j m_j$$  \hspace{1cm} (1.2.45)

Whenever the process in (1.2.45) is well-defined, the magnetization $M$ represents the magnetic dipole moment per unit volume of the medium. The important field vector

$$J_M = \nabla \times M,$$  \hspace{1cm} (1.2.46)

the magnetization current density, can be shown to be the equivalent transport current density that would produce the same magnetic field as $M$ itself and the magnetic
induction due to a magnetized distribution of matter may be computed by taking the curl of the vector potential due to the magnetization, i.e., by taking the curl of

$$A_M(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_\Omega \frac{J_M(\mathbf{r'})}{\| \mathbf{r} - \mathbf{r'} \|} d\mathbf{\bar{r}} \quad (1.2.47)$$

We are now in a position to discuss the appropriate modification, to equations (1.2.35) and (1.2.39), which ensues whenever the magnetic field includes a contribution from a magnetized material. First of all, inasmuch as the field produced by a magnetized medium is derivable from atomic currents, \( B \) can always be written as the curl of a vector potential where, in fact, the vector potential due to the magnetization is given by (1.2.47); thus, equation (1.2.35) still applies. On the other hand, in (1.2.39), we must take care to include all the distinct types of currents that are capable of producing a magnetic field; the correct modification of (1.2.39), therefore, reads

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \mathbf{J}_M) \quad (1.2.48)$$

with \( \mathbf{J} \) the transport current density and \( \mathbf{J}_M \) the magnetization current density. Inserting (1.2.46) into (1.2.48), we readily find that Ampere's law applies, namely,

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (1.2.49)$$

where \( \mathbf{H} \), the magnetic intensity field, is defined by

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad (1.2.50)$$

The basic magnetic field equations now assume the form (1.2.35) and (1.2.49) and must be supplemented by appropriate boundary conditions and an experimentally motivated relationship between \( \mathbf{B} \) and \( \mathbf{H} \); we note that, as an immediate consequence of Stokes' theorem, we obtain from (1.2.49) the fact that

$$\int_S \nabla \times \mathbf{H} \cdot \mathbf{n} \, da = \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot \mathbf{n} \, da \quad (1.2.51)$$

where \( C \) is any space curve bounding the surface \( S \). Equation (1.2.51) implies that the line integral of the tangential component of the magnetic intensity around a closed path \( C \) in \( \mathbb{R}^3 \) is equal to the net transport current through the surface bounded by \( C \). As for the relationships which exist among the field vectors displayed in (1.2.50),
there exist a large class of materials for which an approximate linear relationship holds between $M$ and $H$ and, if the material is isotropic as well as linear, then

$$M = \chi_m H$$  \hspace{1cm} (1.2.52)

where $\chi_m$ is a dimensionless scalar called the magnetic susceptibility; for $\chi_m > 0$, the medium is termed paramagnetic and magnetic induction is strengthened by the presence of the material, while for $\chi_m < 0$ the medium is termed diamagnetic and the presence of the medium weakens magnetic induction. For both paramagnetic and diamagnetic materials, $\chi_m$, which may be temperature dependent, is observed to be very small, i.e., $|\chi_m| \ll 1$. If we introduce the constitutive relation (1.2.52) into (1.2.50), and define the permeability $\mu$ by

$$\mu = \mu_0 (1 + \chi_m)$$  \hspace{1cm} (1.2.53)

then we clearly have a linear relationship between $B$ and $H$, namely,

$$B = \mu H$$  \hspace{1cm} (1.2.54)

However, many materials, the ferromagnets, are characterized by a permanent magnetization and for such materials the magnetization, once established, does not disappear when $H$ is brought to a value of zero; for studying the phenomenon of magnetization in ferromagnetic media, a nonlinear constitutive law for isotropic materials, of the form (1.2.54), but with $\mu = \mu(H)$, has been widely studied. Finally, we comment briefly on the problem of appropriate boundary conditions to be associated with (1.2.35) and (1.2.49). If we have an interface between two media with different magnetic properties, or between a magnetic material and a vacuum, then arguments analogous to those which led to (1.1.28) and (1.1.29), but based now on the magnetic field equations (1.2.35) and (1.2.49), lead to the results

$$(B_1)_n = (B_2)_n; \quad (H_2 - H_1)_t = j \times n$$  \hspace{1cm} (1.2.55)

where $B_i$, $H_i$, $i = 1, 2$ are the magnetic induction and magnetic intensity fields, respectively, in the two different media, the $n$ and $t$ subscripts denote normal and tangential components of the indicated vector fields, $n$ is the exterior unit normal to the interface, and $j$ is the surface current density.
ELECTRIC CURRENTS AND ELECTROMAGNETIC INDUCTION

In the natural scheme of things, in the study of classical electromagnetic theory, we have now arrived at the point where we are able to address the phenomenon of electromagnetic induction. It was observed, principally by Faraday and Henry, that the equation which characterizes electrostatics, i.e. \( \nabla \times \mathbf{E} = 0 \) or, in integral form, \( \oint \mathbf{E} \cdot d\mathbf{l} = 0 \), remains valid provided that the only magnetic force present is that due to a steady current. However, \( \nabla \times \mathbf{E} = 0 \) does not hold in the presence of more general time-dependent fields. It turns out to be useful to define a new quantity, the EMF or \textit{electromotive force}, around a closed circuit \( C \) in \( \mathbb{R}^3 \), by

\[ \mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} \]  

so that \( \mathcal{E} = 0 \) for both static \( \mathbf{E} \) and \( \mathbf{B} \) fields. In the presence of time-dependent fields, however, the electric field \( \mathbf{E} \) can no longer be derived from Coulomb's law but, rather, is now defined so that the Lorentz force, given by (1.2.19), is the electromagnetic force experienced by a test charge \( q \). As a result of experiments, largely associated with Faraday, a relationship of the form

\[ \mathcal{E} = -\frac{d\Phi}{dt} \]  

between the EMF and the change in magnetic flux through a circuit emerged and, indeed, (1.2.57) is known as Faraday's law of electromagnetic induction; it turns out that (1.2.57) is independent of the way in which \( \Phi \) is varied, i.e., independent of the manner in which the values of \( \mathbf{B} \) interior to the circuit are changed. If we employ the definitions of \( \mathcal{E} \) and \( \Phi \), then (1.2.57) assumes the form

\[ \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da \]  

where \( C \) is any space curve in \( \mathbb{R}^3 \) bounding the surface \( S \). If we now assume that \( C \) is a rigid, stationary circuit in \( \mathbb{R}^3 \), then by bringing the time derivative into the surface integral, transforming the line integral via Stokes' theorem, and noting that the resulting integral relation must hold for all fixed bounded surfaces embedded in \( \mathbb{R}^3 \), we are led to the differential form of Faraday's law, namely,

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]  

The negative sign in (1.2.59) is indicative of the fact that the direction of the induced EMF is such as to oppose the change in the magnetic induction field which produces
it; what this means in a practical sense is that if one tries to increase the flux through
a circuit, then the induced EMF causes currents in a direction such as to decrease
the flux, i.e., when we try to insert a pole of a magnet into a coil, the currents
produced by the induced EMF set up a magnetic field which acts to repel the magnet.

Having deduced (1.2.59) from experimental data, we are now in possession of the
third of Maxwell’s four equations and there remains only the task of generalizing the
relation (1.2.49) so as to take into account the presence of time-varying fields; this
generalization of (1.2.49) will be accomplished in the next section. Before proceeding
to complete the full set of Maxwell’s equations, however, we will conclude this section
by looking at the concept of energy density in a magnetic field.

Suppose we consider a set of rigid current-carrying circuits, all of which are
bounded (i.e. lie in a bounded domain of $\mathbb{R}^3$) and embedded in a magnetic medium
which conforms to the linear constitutive hypothesis (1.2.54); the energy of such a
system is easily shown to have the form

$$ E = \frac{1}{2} \sum_{j=1}^{n} I_j \Phi_j $$

(1.2.60)

where $I_j$ is the current in the $j$-th circuit, while $\Phi_j$ is the associated magnetic flux.

For a single circuit $C_j$, we may write that

$$ \Phi_j = \int_{S_j} B \cdot n \, da = \oint_{C_j} A \cdot dl_j $$

(1.2.61)

with $A$ the associated vector potential, so that (1.2.60) assumes the form

$$ E = \frac{1}{2} \sum_{j=1}^{n} \oint_{C_j} I_j A \cdot dl_j $$

(1.2.62)

Now suppose that we are not dealing with currents in circuits defined by physical
wires but, rather, that each circuit in our assumed conducting medium is a closed
path of current density. Employing (1.2.62) for a large number of contiguous $C_j$,
and identifying $I_j dl_j$ with $J \, dv$, a simple limiting argument leads us, for $\Omega \subseteq \mathbb{R}^3$ a
bounded domain, to

$$ E = \frac{1}{2} \int_{\Omega} J \cdot A \, dv $$

(1.2.63)

or, inasmuch as $\nabla \times H = J$, and

$$ \nabla \cdot (A \times H) = H \cdot (\nabla \times A) - A \cdot (\nabla \times H), $$
to
\[ E = \frac{1}{2} \int_H \mathbf{H} \cdot (\nabla \times \mathbf{A}) \, dv - \frac{1}{2} \int_{\partial \Omega} (\mathbf{A} \times \mathbf{H}) \cdot \mathbf{n} \, da \]  \hspace{1cm} (1.2.64)

By allowing \( \partial \Omega \) to move out to infinity, and noting that \( \| \mathbf{A} \| \) decreases at least as fast as \( \frac{1}{r} \), \( r \) being the distance from a point in \( \Omega \) to the origin of whatever coordinate system we have chosen, we easily find that (1.2.64) yields

\[ E = \frac{1}{2} \int_{\mathbb{R}^3} \mathbf{H} \cdot \mathbf{B} \, dv \]  \hspace{1cm} (1.2.65)

thus leading us to define, as the energy density in the magnetic field, the quantity

\[ u_M = \frac{1}{2} \mathbf{H} \cdot \mathbf{B} \]  \hspace{1cm} (1.2.66)

The definition (1.2.66) is the natural counterpart to (1.1.39), which serves to define the energy density in an electrostatic field. In the next section we will generalize these concepts and move on to a consideration of electromagnetic energy in the presence of time varying fields. First, however, we must consider Maxwell’s generalization of Ampere’s law, which is needed in all situations where \( \frac{\partial \mathbf{H}}{\partial t} \neq 0 \).

### 1.3 Maxwell’s Equations and the Propagation of Electromagnetic Waves

In the last section we saw that the magnetic field due to a current distribution satisfied Ampere’s law (1.2.49), the integral form of which is given by (1.2.51); the aforementioned forms of Ampere’s law assume, implicitly, that we are dealing with situations in which the charge density \( \rho \) does not change with time, for (1.2.49) implies that \( \nabla \cdot \mathbf{J} = 0 \) and this is compatible with the equation of continuity (1.2.8) only if \( \frac{\partial \rho}{\partial t} = 0 \).

For physical situations in which \( \frac{\partial \rho}{\partial t} \neq 0 \), Ampere’s circuital law must be generalized; the appropriate generalization of Ampere’s law, an achievement due to Maxwell, consists of adding a displacement current \( \frac{\partial \mathbf{D}}{\partial t} \) to the right-hand side of Ampere’s law (1.2.49). In fact, inasmuch as \( \nabla \cdot \mathbf{D} = \rho \) we have, by virtue of the equation of continuity (1.2.8), that

\[ \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{D}) = \nabla \cdot \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0 \]  \hspace{1cm} (1.3.1)
In view of (1.3.1), it is clear that Ampere's law, for situations in which the fields are not "slowly varying" (so that the displacement current $\frac{\partial D}{\partial t}$ cannot be ignored in comparison with the transport current $J$) must be modified to read

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$  \hspace{1cm} (1.3.2)

We are now in possession of the entire set of Maxwell's equations, namely, the extension of Ampere's law (1.3.2), Faraday's law of electromagnetic induction (1.2.59), Gauss's law (1.1.25), and (1.2.35), which remains valid for time-varying fields and is the mathematical expression of the fact that isolated magnetic poles have not been observed (at least not as of the time of preparation of this manuscript). For the sake of convenient future reference, we collect below the full set of Maxwell's equations and renumber them accordingly; it is important to state explicitly that these equations are the mathematical expressions of a vast collection of experimental results and observations and, as such, are not susceptible to mathematical proof. However, Maxwell's equations are the basic equations governing the classical electromagnetic fields produced by source charges and current densities and, as with basic field equations in mechanics, such as conservation of mass or balance of momentum, are thought to apply to all macroscopic electromagnetic phenomena. In summary, the fundamental equations are

$$\nabla \cdot D = \rho$$  \hspace{1cm} (1.3.3a)

$$\nabla \cdot B = 0$$  \hspace{1cm} (1.3.3b)

$$\nabla \times E = -\frac{\partial B}{\partial t}$$  \hspace{1cm} (1.3.3c)

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$  \hspace{1cm} (1.3.3d)

and in the presence of material bodies we must also prescribe constitutive relations among the field vectors appearing above of the form, e.g., $D = D(E)$, $H = H(B)$, and $J = J(E)$. There is no need to add the equation of continuity to the set, as (1.2.8) follows as a direct consequence of (1.3.3d) and (1.3.3a). Once $E$ and $B$ have been determined (at least in principle) from the field equations (1.3.3a)-(1.3.3d), an appropriate set of constitutive relations, and whatever initial and boundary conditions are germane to a particular physical problem, and an associated geometry, then the
Lorentz force equation (1.2.19) is applicable and serves to describe the action of the electromagnetic field on charged particles.

We now consider the matter of electromagnetic energy in the presence of time-varying fields. In the two previous sections, we have noted that the electrostatic potential energy may be expressed in the form

\[ E_E = \frac{1}{2} \int \nabla \cdot \Phi \, d\mathbf{r} \quad (1.3.4) \]

with an analogous expression for the energy stored in a magnetic field, namely,

\[ E_M = \frac{1}{2} \int \nabla \cdot \Phi \, d\mathbf{r} \quad (1.3.5) \]

For time-varying fields, the relevant equations are those of Maxwell (1.3.3a)-(1.3.3d). If we take the dot product of (1.3.3d) with \( \mathbf{E} \) and subtract the equation produced from the result of taking the dot product of (1.3.3c) with \( \mathbf{H} \), we obtain

\[ \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J} \quad (1.3.6) \]

However, for sufficiently smooth vector fields \( \mathbf{R}, \mathbf{S} \), we know that

\[ \nabla \cdot (\mathbf{R} \times \mathbf{S}) = \mathbf{R} \cdot (\nabla \times \mathbf{S}) - \mathbf{S} \cdot (\nabla \times \mathbf{R}) \quad (1.3.7) \]

and an application of this vector identity to (1.3.6) yields

\[ \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J} \quad (1.3.8) \]

When the substance to which we want to apply (1.3.8) is both linear and nondispersive, so that \( \mathbf{D} = \epsilon \mathbf{E}, \mathbf{B} = \mu \mathbf{H} \), with \( \epsilon, \mu \) time-independent, as well as independent of the field vectors \( \mathbf{E} \) and \( \mathbf{H} \), respectively, then

\[ \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \mathbf{E} \cdot \frac{\partial}{\partial t} (\epsilon \mathbf{E}) = \frac{1}{2} \epsilon \frac{\partial}{\partial t} \| \mathbf{E} \|^2 = \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{E} \cdot \mathbf{D} \right) \]

\[ \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{H} \cdot \frac{\partial}{\partial t} (\mu \mathbf{H}) = \frac{1}{2} \mu \frac{\partial}{\partial t} \| \mathbf{H} \|^2 = \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{H} \cdot \mathbf{B} \right) \]

and (1.3.8) assumes the form

\[ \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t} \left( \mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H} \right) - \mathbf{J} \cdot \mathbf{E} \quad (1.3.9) \]

Comparing the expression on the right-hand side of (1.3.9) with the integrands in (1.3.4) and (1.3.5), we see that it is still appropriate in this case to identify \( \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \).
\( H \cdot B \) as being the sum of the electric and magnetic energy densities; the physical interpretation of the term \( J \cdot E \) is that it represents, in most instances (e.g., for \( J = \sigma E \)), the negative of the *Joule heating rate* per unit volume.

If we integrate (1.3.9) over a bounded domain \( \Omega \subseteq \mathbb{R}^3 \), with sufficiently smooth boundary \( \partial \Omega \), and then apply the divergence theorem to the left-hand side of the resulting identity, we obtain

\[
- \int_{\Omega} J \cdot E \, dv = \frac{d}{dt} \int_{\Omega} \frac{1}{2} [E \cdot D + B \cdot H] \, dv \\
+ \oint_{\partial \Omega} E \times H \cdot n \, da \tag{1.3.10}
\]

Equation (1.3.10) shows that \( \int_{\Omega} J \cdot E \, dv \) is composed of two parts: the rate of change of the electromagnetic energy stored in \( \Omega \) plus a surface integral representing the rate of energy flow across the bounding surface \( \partial \Omega \); with this interpretation (1.3.10) expresses conservation of electromagnetic energy in a fixed volume \( \Omega \subseteq \mathbb{R}^3 \). If in (1.3.9) we now set

\[
S^p = E \times H \tag{1.3.11}
\]

and

\[
u \equiv u_E + u_M = \frac{1}{2} (E \cdot D + B \cdot H) \tag{1.3.12}
\]

then from (1.3.9) we obtain directly that

\[
\nabla \cdot S^p + \frac{\partial u}{\partial t} = -J \cdot E \tag{1.3.13}
\]

If \( \nabla \cdot S^p = 0 \) in (1.3.13), then the resulting relation expresses local conservation of energy, i.e., at any point \( \mathbf{r} \in \Omega \subseteq \mathbb{R}^3 \), the rate of change of the electromagnetic field energy equals the power dissipation per unit volume. If, however, \( J \cdot E = 0 \) but \( \nabla \cdot S^p \neq 0 \) (e.g. in a nonconducting medium), then (1.3.13) reduces to

\[
\nabla \cdot S^p + \frac{\partial u}{\partial t} = 0 \tag{1.3.14}
\]

which is of the same form as the equation of continuity (1.2.8) except that \( S^p \) now assumes the role previously taken by \( J \), while the energy density of the electromagnetic field \( u \) assumes the role of the charge density \( \rho \); in situations where (1.3.14) applies, \( \nabla \cdot S^p \) then represents the divergence of an energy current density or, equivalently,
a rate of energy flow per unit area. The vector $\mathbf{S}^p$, defined by (1.3.11), is known in electromagnetic theory as the Poynting vector.

Probably the most important of all consequences of Maxwell’s equations are the equations governing electromagnetic wave propagation; while the relevant wave equations are presented in most texts on the subject only for wave propagation in a linear medium, we will present here both the usual linear wave equation and a strongly coupled system of nonlinear wave equations for the components $D_i(x, t)$ of the electric displacement field, in a nonlinear dielectric medium which conforms to a constitutive hypothesis of the form

$$\begin{cases} 
D(x, t) = D(E(x, t)) \\
B(x, t) = \mu H(x, t) 
\end{cases} \tag{1.3.15}$$

with $D(0) = 0$, $\mu > 0$ (constant), and

$$\det \left[ \frac{\partial D_i}{\partial E_j} \right]_{E=0} \neq 0 \tag{1.3.16}$$

In a linear medium, with $D = \epsilon E$, and $J = \sigma E$ ($\epsilon, \sigma > 0$ and constant) we first take the curl on both sides of (1.3.3d), and substitute from the constitutive relations, so as to obtain

$$\nabla \times \nabla \times H = \sigma \nabla \times E + \epsilon \frac{\partial}{\partial t}(\nabla \times E) \tag{1.3.17}$$

Employing (1.3.3c) in (1.3.17), together with the constitutive hypothesis $B = \mu H$, $\mu > 0$ and constant, we find that

$$\nabla \times \nabla \times H = -\sigma \mu \frac{\partial H}{\partial t} - \epsilon \mu \frac{\partial^2 H}{\partial t^2} \tag{1.3.18}$$

However, for any sufficiently smooth vector field $\mathbf{v}$, we have the well-known identity

$$\nabla \times \nabla \times \mathbf{v} = \nabla \nabla \cdot \mathbf{v} - \nabla^2 \mathbf{v}$$

whose use in (1.3.18) readily yields

$$\nabla \nabla \cdot H - \nabla^2 H = -\sigma \mu \frac{\partial H}{\partial t} - \epsilon \mu \frac{\partial^2 H}{\partial t^2} \tag{1.3.19}$$

or, as $\nabla \cdot H = \frac{1}{\mu} \nabla \cdot B = 0$, by virtue of (1.3.3b), the wave equation

$$\epsilon \mu \frac{\partial^2 H}{\partial t^2} + \sigma \mu \frac{\partial H}{\partial t} = \nabla^2 H \tag{1.3.20}$$
for the evolution of the magnetic intensity field $H$. If we begin a similar reduction by first taking the curl on both sides of the third of Maxwell's equations, (1.3.3c), we are easily led to the same wave equation (1.3.20) for the evolution of the electric field vector $E$, namely,

$$
\epsilon \mu \frac{\partial^2 E}{\partial t^2} + \sigma \mu \frac{\partial E}{\partial t} = \nabla^2 E
$$

(1.3.21)

provided $\nabla \cdot D = 0$ in the linear medium, that is, provided the charge density is zero.

Inasmuch as (1.3.21) applies only to linear media, in which the charge density $\rho \equiv 0$, and the only current density $J$ is that which arises from the passive response of the medium to the electric field of the wave, we now want to consider (still for a linear medium) the situation in which there are prescribed charge and current distributions $\rho(x, t)$ and $J(x, t)$, respectively; the standard approach here consists of noting that, as the magnetic induction field $B$ is divergence free, there exists a vector potential $A(x, t)$ such that (1.2.41) is satisfied at each point $x \in \Omega \subseteq \mathbb{R}^3$ and each $t \geq 0$. Employing (1.2.41) in Maxwell's third equation (1.3.3c), and interchanging the spatial and temporal differentiations, we are led to the equation

$$
\nabla \times \left( \frac{\varepsilon E}{\varepsilon} + \frac{\partial A}{\partial t} \right) = 0
$$

(1.3.22)

But (1.3.22) implies that there exists a scalar-valued function $\phi(x, t)$ such that

$$
E = -\nabla \phi - \frac{\partial A}{\partial t}
$$

(1.3.23)

so that (1.2.41) and (1.3.23) now yield both the electric and magnetic fields in terms of a vector potential $A$ and a scalar potential $\phi$. By substituting (1.2.41) and (1.3.23) into the fourth of Maxwell's equations, (1.3.3d), after first setting $D = \varepsilon E$ and $H = \frac{1}{\mu} B$, we are led to a wave equation for $A$,

$$
-\nabla^2 A + \varepsilon \mu \frac{\partial^2 A}{\partial t^2} + \nabla \nabla \cdot A + \varepsilon \mu \nabla \left( \frac{\partial \phi}{\partial t} \right) = \mu J,
$$

(1.3.24)

where we have used the aforementioned vector identity for $\nabla \times \nabla \times A$. Although $\nabla \times A = B$ is specified, the choice of $\nabla \cdot A$ has, until this point, been arbitrary. If we now impose the Lorentz condition, namely,

$$
\nabla \cdot A = -\varepsilon \mu \frac{\partial \phi}{\partial t}
$$

(1.3.25)
then, clearly, (1.3.24) reduces to the wave equation
\[ \epsilon \mu \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu \mathbf{J} \] (1.3.26)

We next return to the representation of \( \mathbf{E} \) given by (1.3.23) and substitute this relation in Maxwell's first equation (1.3.3a) after setting \( \mathbf{D} = \epsilon \mathbf{E} \); this substitution yields as a consequence the equation
\[ -\epsilon \left( \nabla \cdot \nabla \phi + \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} \right) = \rho, \] (1.3.27)
where \( \rho = \rho(\mathbf{x}, t) \) is the prescribed charge density. Finally, if we write \( \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) \) in (1.3.27), and employ the Lorentz condition (1.3.25), we obtain for the scalar potential \( \phi \) a wave equation entirely analogous to (1.3.26), namely,
\[ \epsilon \mu \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{1}{\epsilon} \rho \] (1.3.28)

The solutions of the inhomogeneous wave equations (1.3.26), (1.3.28), for prescribed fields \( \rho, \mathbf{J} \), and associated initial and boundary data, consist, of course, of the general solution of the associated homogeneous linear wave equations plus particular solutions of the inhomogeneous equations; for the inhomogeneous wave equations (1.3.26), (1.3.28), particular solutions are well-known and are given by the \textit{retarded scalar and vector potentials}

\[ \phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon} \int_{\Omega} \frac{\rho(\tilde{\mathbf{x}}, \tilde{t})}{||\mathbf{x} - \tilde{\mathbf{x}}||} d\tilde{v} \] (1.3.29a)
\[ \mathbf{A}(\mathbf{x}, t) = \frac{\mu}{4\pi} \int_{\Omega} \frac{\mathbf{J}(\tilde{\mathbf{x}}, \tilde{t})}{||\mathbf{x} - \tilde{\mathbf{x}}||} d\tilde{v} \] (1.3.29b)

where \( \tilde{t} = t - \sqrt{\epsilon\mu}||\mathbf{x} - \tilde{\mathbf{x}}|| \) is the retarded time. Once the scalar and vector potentials \( \phi \) and \( \mathbf{A} \) have been computed, the fields \( \mathbf{B} \) and \( \mathbf{E} \) may be determined from (1.2.41) and (1.3.23), respectively. Of course the fact that \( \mathbf{A} \) and \( \phi \) satisfy the inhomogeneous wave equations (1.3.26) and (1.3.28) is due to our imposition of the Lorentz condition (1.3.25); however, if a particular choice of the potentials \( \mathbf{A} \) and \( \phi \) yield, through (1.2.41) and (1.3.23), the appropriate electric and magnetic fields, then a \textit{gauge transformation} of the form

\[
\begin{align*}
\hat{\mathbf{A}} &= \mathbf{A} + \nabla \eta \\
\hat{\phi} &= \phi - \frac{\partial \eta}{\partial t}
\end{align*}
\] (1.3.30)
for arbitrary sufficiently smooth $\eta(x, t)$, yields new potentials which produce the same $E$ and $B$ fields. If we substitute from (1.3.30) into (1.3.24), with $(\hat{A}, \hat{\phi})$ replacing $(A, \phi)$, we readily find that

$$\nabla^2 \eta - \varepsilon\mu \frac{\partial^2 \eta}{\partial t^2} = -\left(\nabla \cdot A + \varepsilon\mu \frac{\partial \phi}{\partial t}\right) = 0 \quad (1.3.31)$$

so that $(\hat{A}, \hat{\phi})$ satisfy the Lorentz condition, whenever $(A, \phi)$ do, provided that $\eta$ satisfies the usual scalar wave equation. Even if $(A, \phi)$ did not satisfy the Lorentz condition, it is clear from (1.3.31) that $(\hat{A}, \hat{\phi})$ will, provided the scalar-valued function $\eta$ in (1.3.30) satisfies an inhomogeneous wave equation.

At this point it is worthwhile to comment on one very important aspect of the linear wave equations (1.3.26) and (1.3.28), which were deduced from Maxwell's equations (1.3.3a)-(1.3.3d) under the assumptions that the charge density $\rho$ in the medium (either conducting or nonconducting) is zero and that the only current density $J$ is the one which arises in passive response to the electric field in the wave: while (1.3.20) and (1.3.21) are consequences, in the aforementioned situation, of Maxwell's equations, the converse does not follow and Maxwell's equations (1.3.3a)-(1.3.3d) must still serve as a restriction on solutions of the wave equations (1.3.20) and (1.3.21).

The boundary conditions, at an interface between two media, are derived from Maxwell's equations, just as in the static case, using the familiar pillbox-shaped surfaces at the interface. From (1.3.3b) we readily deduce, just as in the static case, that $(B_1 - B_2) \cdot n = 0$, where $B_i$ is the limiting value of $B$ in the $i$-th medium as we approach a point on the interface, while $n$ is the exterior unit normal to the interface. Also, from (1.3.3c) it follows that $(E_1 - E_2)_t$ is proportional to

$$\oint_S \frac{\partial B}{\partial t} \cdot n \, da$$

where the subscript $t$ denotes the tangential component of the indicated vector field on the interface, while $S$ denotes the surface bounded by an infinitesimal rectangular path intersecting the interface. If $\frac{\partial B}{\partial t}$ is bounded in a neighborhood of the interface, then the above integral vanishes as the sides of the rectangular path are shrunk down to zero length and we again find, as in the static case, that $(E_1)_t = (E_2)_t$. From the first of Maxwell's equations (1.3.3a), which we also integrate over a pillbox-shaped domain intersecting the interface, we find that $(D_1 - D_2) \cdot n = \sigma_s$, where $\sigma_s$ is the
surface charge density on the interface; an analogous procedure, when applied to the
equation of continuity (1.2.8), produces the fact that $(J_1 - J_2) \cdot n = -\frac{\partial \sigma_s}{\partial t}$. Finally, if we integrate the last of Maxwell’s equations (1.3.3d) over the same rectangular type of path that was employed to derive the boundary condition on $E_t$ at the interface, we find that $(H_1 - H_2)_t = 0$, except in those cases where the conductivity $\sigma$, which appears in the constitutive relation (1.2.9), is infinite.

While we do not wish to dwell, in this monograph, on the matter of wave propagation in linear media, a brief discussion of special solutions of the wave equations (1.3.20), (1.3.21) should serve as a useful adjunct to the analysis of nonlinear wave propagation which will follow in the next two chapters. The class of solutions of the linear wave equations (1.3.20) and (1.3.21) which is most amenable to an elementary analysis consists of the so-called **monochromatic waves**, which are characterized by a single frequency. For monochromatic waves, one may obtain a solution of the wave equation (1.3.21) for $E$ in order to compute $\nabla \times E$ which, in turn, will yield $\frac{\partial B}{\partial t}$; for a monochromatic wave the relationship between $B$ and $\frac{\partial B}{\partial t}$ is then sufficiently simple, so that a solution of (1.3.20) will follow directly. Thus, if we look for solutions of (1.3.21) of the form

$$E(x, t) = E(x)e^{-i\omega t},$$  \hspace{1cm} (1.3.32)

where the physically relevant electric field is obtained by taking the real part of (1.3.32), $E(x)$, in general, also being complex-valued, then substitution in (1.3.21) yields, for the spatial part of $E$, the equation

$$\nabla^2 E + \omega^2 \varepsilon \mu E + i\omega \mu E = 0$$  \hspace{1cm} (1.3.33)

Several simple cases of (1.3.33) are commonly encountered in practice: if the wave is propagating in empty space, then $\sigma = 0$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$. If, in addition, $E(x)$ varies in only one space dimension, say the $x$-direction, then $E(x) = (E_1(x), E_2(x), E_3(x))$ and (1.3.33) reduces to

$$\frac{d^2 E(x)}{dx^2} + \left(\frac{\omega}{c}\right)^2 E(x) = 0$$  \hspace{1cm} (1.3.34)

with $c = \sqrt{\frac{1}{\varepsilon_0 \mu_0}}$ the speed of light in a vacuum (indeed, this consequence of Maxwell’s equations confirmed the electromagnetic nature of light). The Helmholtz equation
(1.3.34) has solutions of the form

\[ E(x) = \hat{E} e^{\pm i \kappa x}, \]  

(1.3.35)

with \( \hat{E} \) a constant vector, provided that the wave number \( \kappa = \omega / c \). Combining (1.3.32) with (1.3.35), we have the following solution of (1.3.21) in this special case, namely,

\[ E(x, t) = \hat{E} e^{-i(\omega t \pm \kappa x)} \]  

(1.3.36)

whose real part

\[ \text{Re} \, E(x, t) = \hat{E} \cos(\omega t \pm \kappa x) \]  

(1.3.37)

represents a sinusoidal wave traveling either to the right or left in the \( x \)-direction with speed \( \frac{\omega}{\kappa} = c \). From (1.3.37) we gather that the wave has frequency \( f = \frac{\omega}{2\pi} \) and wavelength \( \lambda = \frac{2\pi}{\kappa} \), so that \( \lambda f = c \).

If the medium is a nonconducting, linear, nonmagnetic dielectric then, again, \( \sigma = 0, \mu = \mu_0 \), but \( \epsilon = K \varepsilon_0 \), where \( K \) is termed the dielectric constant; recall that, in general, \( D = \epsilon E \), with the permittivity of the form (1.1.27), and for linear media the susceptibility \( \chi \) is, most often, a constant so that

\[ K = \frac{\epsilon}{\varepsilon_0} = 1 + \frac{\chi}{\varepsilon_0} \]  

(1.3.38)

is constant. In this case the same results hold as in the vacuum situation, but with \( \kappa = \frac{\sqrt{K \omega}}{c} \), and the velocity of the propagating wave is now \( c/n \) instead of \( c \), where \( n = \sqrt{\kappa} \) is termed the index of refraction of the linear dielectric medium. If, on the other hand, the medium is conducting, then \( \sigma > 0 \) and the last term in (1.3.33) cannot be dropped; two extreme subcases which arise here are those in which either \( \sigma \ll \omega \epsilon \) or \( \sigma \gg \omega \epsilon \). For \( \sigma \gg \omega \epsilon \) it is common to ignore the second term on the left-hand side of (1.3.33), in which case, again for a one spatial-dimension dependence assumed for \( E \), we obtain

\[ \frac{d^2 E(x)}{dx^2} + i \omega \sigma \mu E(x) = 0 \]  

(1.3.39)

If we assume a purely imaginary frequency \( \omega \), so that \( \gamma = i \omega \) is real, and set \( \kappa = \sqrt{\gamma \sigma \mu} \), then the same spatial dependence for \( E(x) \) as that obtained in (1.3.35) applies in this situation as well, but now, in place of (1.3.36), we obtain

\[ E(x, t) = \hat{E} e^{\pm i \kappa x} e^{-\gamma t} \]  

(1.3.40)
so that the electric field in the wave, instead of oscillating, decays exponentially in time.

Finally, we want to consider general plane wave solutions of (1.3.33), beginning first with the case of plane monochromatic waves in nonconducting media and then proceeding to the case of conducting media. Recall that a plane wave propagating in the direction \( \mathbf{l} \) is described by the function \( \exp\{-i(\omega t - \kappa \mathbf{l} \cdot \mathbf{x})\} \) and if we define a propagation vector \( \kappa = \kappa \mathbf{l} \), then we may write this as \( \exp\{-i(\omega t - \kappa \cdot \mathbf{l})\} \). In the special cases considered above, we took \( \mathbf{l} = \mathbf{i} \), the unit vector in the \( x \) direction, so that \( \mathbf{l} \cdot \mathbf{z} = x \). By definition, the velocity of propagation of a plane monochromatic wave is just the velocity that the planes of constant phase move with, where by constant phase we mean that \( \kappa \cdot \mathbf{l} - \omega t \) = constant. As \( \kappa \cdot \mathbf{l} = ||\kappa||||\mathbf{l}|| \cos \theta \equiv \kappa \tilde{l} \), \( \kappa = ||\kappa|| \) and \( \tilde{l} \) is the projection of \( \mathbf{l} \) on the direction of \( \kappa \), we have for the phase velocity \( v_p \)

\[
v_p = \frac{d\tilde{l}}{dt} = \frac{\omega}{\kappa} \tag{1.3.41}
\]

Now, in order to obtain the plane wave solutions \( E, B \) of the wave equations (1.3.20) and (1.3.21), it actually turns out to be more convenient to go back to Maxwell’s equations (1.3.3a)-(1.3.3d); in these equations, as we are first dealing with a nonconductor, and will assume that there are no prescribed charge or current densities, we have \( \rho = 0 \) in (1.3.3a) and \( J = 0 \) in (1.3.3d). We begin by assuming that the electric field has the form

\[
E(x, t) = \hat{E} \exp\{-i(\omega t - \kappa \cdot \mathbf{l})\} \tag{1.3.42}
\]

with analogous expressions for \( D, B, \) and \( H \); upon substituting \( E \) from (1.3.42), and the similar expressions for \( D, B, \) and \( H \), into (1.3.3a)-(1.3.3d), with \( \rho = 0, J = 0 \), we obtain the following set of four equations to be satisfied by \( \hat{E}, \hat{D}, \hat{B}, \) and \( \hat{H} \), the (constant) complex vector amplitudes associated with the plane wave:

\[
\kappa \times \hat{E} = \omega \hat{B} ; \quad \kappa \cdot \hat{D} = 0 \tag{1.3.43}
\]

\[
\kappa \cdot \hat{B} = 0 \quad ; \quad \kappa \times \hat{H} = -\omega \hat{D} \tag{1.3.44}
\]

where, as a direct consequence of the linearity of the nonconducting medium,

\[
\hat{D} = \epsilon \hat{E} \quad ; \quad \hat{H} = \frac{1}{\mu} \hat{B} \tag{1.3.45}
\]
The $\varepsilon$ and $\mu$ are assumed here to be constant scalars, so that the medium is also both isotropic and homogeneous and, if we also take the medium to be nonmagnetic, then we may as well set $\mu = \mu_0$. Noting again that $\varepsilon = K\varepsilon_0$, while $\varepsilon_0\mu_0 = 1/c^2$, we easily find that as a consequence of (1.3.43)-(1.3.45) we have

$$K\kappa \cdot \hat{E} = 0 \quad ; \quad \kappa \times \hat{E} = \omega \hat{B} \quad (1.3.46)$$

$$\kappa \cdot \hat{B} = 0 \quad ; \quad \kappa \times \hat{B} = -\frac{\omega}{c^2} K \hat{E} \quad (1.3.47)$$

which for a given pair $(\omega, K)$ is a set of (vector) algebraic equations to be satisfied by $\kappa, \hat{E},$ and $\hat{B}$. If $K \neq 0$, then from the first equation in each of the sets (1.3.46), (1.3.47) we see that both $\hat{E}$ and $\hat{B}$ must be orthogonal to $\kappa$ and a plane wave with this property is termed *transverse*. But, from the second equation in the set (1.3.46), we have that $\hat{B}$ is proportional to $\kappa \times \hat{E}$ so that $\hat{E} \cdot \hat{B} = 0$ and, thus, the electric and magnetic field vectors in any such propagating plane wave are also orthogonal to each other and $\kappa, \hat{E}, \hat{B}$ form a (right-handed) orthogonal set. From the second equation in (1.3.46) it follows that $||\hat{B}|| = (\kappa/\omega)||\hat{E}||$; to compute $||\kappa||$ we employ the second equation in each of the sets (1.3.46) and (1.3.47) and find that

$$\kappa \times (\kappa \times \hat{E}) = \omega \kappa \times \hat{B} = -K \left(\frac{\omega}{c}\right)^2 \hat{E} \quad (1.3.48)$$

which, when coupled with the vector identity $\kappa \times (\kappa \times \nu) = (\kappa \cdot \nu)\kappa - \kappa^2 \nu$, and the fact that $\kappa \cdot \hat{E} = 0$, in a transverse wave, yields

$$-K \left(\frac{\omega}{c}\right)^2 \hat{E} = -\kappa^2 \hat{E} \quad (1.3.49)$$

From (1.3.49) it is immediate that

$$\kappa = ||\kappa|| = \frac{\sqrt{K\omega}}{c} \quad (1.3.50)$$

which is often termed the *transverse dispersion relation*.

Now we turn our attention to the case of plane monochromatic waves in a conducting medium, again assuming that there are no prescribed charge or current distributions, but allowing for an induced current density of the form $J = \sigma E$, $\sigma > 0$ constant, which arises in response to the electric field in the wave. Our calculations are entirely analogous to those displayed above for a nonconducting medium, except that we now employ (1.3.3d) with $J = \sigma E$; we are led to

$$\kappa \times \hat{H} = -\omega \hat{D} - i\sigma \hat{E} \quad (1.3.51)$$
in lieu of the second equation in (1.3.44). Substituting, again, \( \dot{H} = \frac{1}{\mu} \dot{B}, \dot{D} = \epsilon \dot{E} \), with \( \epsilon = K \epsilon_0 \), we obtain from (1.3.51) the relation

\[
\kappa \times \dot{B} = -\frac{\omega}{c^2} \left( K + i \frac{\sigma}{\epsilon_0 \omega} \right) \dot{E}
\]  

so that if we define the complex dielectric constant \( K_c \) by

\[
K_c = K + i \frac{\sigma}{\epsilon_0 \omega}
\]  

then

\[
\kappa \times \dot{B} = -\frac{\omega}{c^2} K_c \dot{E}
\]  

which is analogous to the second equation in the set (1.3.47). With the assumptions \( K_c \neq 0 \), and \( \kappa \cdot \dot{E} = 0 \), a transverse dispersion relation similar to (1.3.40) results, namely, \( \kappa = \sqrt{\frac{K_c \omega}{c^2}} \) which, in turn, leads to the definition of a complex refractive index \( n_c = \sqrt{K_c} \). Now, in order to satisfy the relation \( \kappa = n_c \omega/c \), clearly, either \( \kappa \) must be a complex-valued vector or \( \omega \) must be a complex-valued scalar. Suppose that \( \omega \) is real, while \( \kappa = \kappa_R + i \kappa_c \). Then, formally, we have plane wave solutions of Maxwell's equations (1.3.3a)-(1.3.3d) of the form

\[
\begin{align*}
E(x, t) &= \hat{E} e^{-\kappa_c \cdot x} e^{-i(\omega t - \kappa_R \cdot x)} \\
B(x, t) &= \hat{B} e^{-\kappa_c \cdot x} e^{-i(\omega t - \kappa_R \cdot x)}
\end{align*}
\]  

The relations (1.3.55) represent a plane wave which propagates in the direction \( \kappa_R \) with wavelength \( \lambda = \frac{2\pi}{|\kappa_R|} \) and whose amplitude is exponentially decreasing, with the most pronounced decrease occurring if \( x \) is parallel to \( \kappa_c \). Surfaces of constant phase and constant amplitude may be defined, respectively, as planes orthogonal to the directions of \( \kappa_R \) and \( \kappa_c \). From the decomposition of \( \kappa \) into real and complex parts we have

\[
\kappa = \left( \|\kappa_R\|^2 - \|\kappa_c\|^2 + 2i \kappa_R \cdot \kappa_c \right)^{1/2}
\]  

Since \( \kappa = n_c \omega/c \), we may set \( n_c = m + ip \) and proceed to investigate the phase velocity of the wave and the manner in which the amplitude is attenuated in space; however, we will do this in one particularly simple case only, i.e., for the case in which \( \kappa_R \) is parallel to \( \kappa_c \); in this case,

\[
\kappa = \left( \|\kappa_R\| + i \|\kappa_R\| \right) v = \|\kappa\| v
\]
with \( \mathbf{v} \) a unit vector (real) in the direction of \( \mathbf{\kappa}_R \) (and \( \mathbf{\kappa}_c \)). As \( \dot{\mathbf{E}} \cdot \mathbf{v} = \dot{\mathbf{B}} \cdot \mathbf{v} = 0 \), the electric and magnetic field vectors are perpendicular to the propagation direction \( \mathbf{v} \) but \( \dot{\mathbf{B}} = \frac{n_\mathbf{c}}{c} \mathbf{v} \times \dot{\mathbf{E}} \), with \( n_\mathbf{c} \) complex-valued, so that \( \mathbf{E} \) and \( \mathbf{B} \) will not be in phase with one another. With \( n_\mathbf{c} = m + ip \), we easily find that \( ||\mathbf{\kappa}_R|| = m\omega/c \), and \( ||\mathbf{\kappa}_c|| = \rho\omega/c \), and if we set \( \mathbf{v} \cdot \mathbf{x} = \zeta \), then in this case we obtain

\[
\mathbf{E}(\mathbf{x}, t) = \hat{\mathbf{E}} e^{-\rho \omega \zeta/c} e^{-i\omega(t-m\zeta/c)}
\]  (1.3.57)

so that the plane wave propagates with phase velocity \( c/m \) and the attenuation constant is just \( \rho\omega/c \). We note in passing that \( \delta = c/\rho\omega \), the reciprocal of the attenuation constant, is called the skin depth; it is, by (1.3.57), the distance that the wave must propagate into the conducting medium in order for its magnitude to fall off to \( 1/e \) of the value that it had at the bounding surface where the wave entered the medium.

For a nonconducting medium, \( \rho = 0 \) and the skin depth \( \delta = \infty \). Except for the special case just discussed, i.e. when \( \mathbf{\kappa}_R \) and \( \mathbf{\kappa}_c \) are not parallel, \( \mathbf{v} \) must, in general, be taken to be complex and both the phase velocity and attenuation constant will depend on \( m \) and \( p \) (the so-called optical constants) in a rather complicated fashion.

Although we will not begin, in earnest, the study of wave propagation in nonlinear electromagnetic media until we embark on our work in Chapter 2, we deem it feasible, at this juncture, to record here a general nonlinear wave equation for the evolution of the components of the electric displacement vector \( \mathbf{D} \) in a nonlinear dielectric which conforms to the general constitutive hypothesis (1.2.26); if nothing else, making note of such a result here will give the reader a glimpse into the complexity which nonlinearity introduces into the relevant wave equations. The specific result we have in mind is the content of the following elementary

**Lemma 1.1** [25] Let \( \Omega \subseteq \mathbb{R}^3 \) be either a bounded or unbounded domain which is filled with a rigid nonlinear dielectric substance that conforms to the constitutive hypothesis

\[
\mathbf{D} = \epsilon(||\mathbf{E}||) \mathbf{E} ; \quad \mathbf{H} = \mu^{-1} \mathbf{B} ; \quad \mathbf{J} = \sigma(||\mathbf{E}||) \mathbf{E}
\]  (1.3.58)

where \( \epsilon, \sigma \) are differentiable, \( \epsilon, \sigma > 0 \), with \( (\epsilon(\zeta)\zeta)' > 0 \), \( \forall \zeta > 0 \). Then \( \exists \lambda, \eta, \) both differentiable maps of \( \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), such that for \( i = 1, 2, 3 \):

\[
\mu \frac{\partial^2 D_i}{\partial t^2} + \frac{\partial}{\partial t} (\eta(||\mathbf{D}|| D_i)) =
\]
\[ \nabla^2 (\lambda \left\| D \right\| D_i) - \frac{\partial}{\partial x_i} (\nabla \lambda (\left\| D \right\|) \cdot D) \] (1.3.59)

**Proof:** By (1.3.58), \( \| D \| = \epsilon(\| E \|) \| E \| \), so if \((\epsilon(\zeta) \zeta)' > 0\), \( g(\| E \|) \equiv \epsilon(\| E \|) \| E \| \) is invertible and \( \exists g^{-1} : R^+ \to R^+ \) such that \( \| E \| = g^{-1}(\| D \|) \). Using (1.3.58), once again, we find that

\[ E = \frac{1}{\epsilon(\| E \|)} D = \frac{1}{\epsilon(g^{-1}(\| D \|))} D \equiv \lambda(\| D \|) D \] (1.3.60)

Applying the identity

\[ \Delta v = \text{grad}(\nabla \cdot v) - \text{curl curl } v \]

to \( v = E \) (assuming, of course, that \( E \) is sufficiently smooth) we have for \( i = 1, 2, 3 \)

\[ \nabla^2 E_i = \frac{\partial}{\partial x_i} (\nabla \cdot E) - (\text{curl curl } E)_i \] (1.3.61)

Using the third of Maxwell's equations (1.3.3c), then the constitutive hypothesis (1.3.58), and finally the fourth Maxwell equation (1.3.3d), we find, in succession, that

\[
\text{curl curl } E = -\text{curl } \frac{\partial B}{\partial t} \\
= -\mu \text{curl } \frac{\partial H}{\partial t} \\
= -\mu \frac{\partial}{\partial t} (\text{curl } H) \\
= -\mu \left[ \frac{\partial^2 D}{\partial t^2} + \frac{\partial J}{\partial t} \right]
\] (1.3.62)

But

\[
\frac{\partial}{\partial t} J(E) = \frac{\partial}{\partial t} \{\sigma(\| E \|) E\} \\
= \frac{\partial}{\partial t} \{\sigma (\lambda(\| D \|) \| D \|) \lambda(\| D \|) D\} \\
= \frac{1}{\mu} \frac{\partial}{\partial t} \{\eta(\| D \|) D\}
\]

where we have set

\[ \eta(\zeta) = \mu \sigma(\lambda(\zeta) \zeta) \lambda(\zeta), \quad \zeta \geq 0 \] (1.3.64)
Combining (1.3.61)-(1.3.63) we have
\[ \nabla^2 E_i = \frac{\partial}{\partial x_i} (\nabla \cdot E) + \mu \frac{\partial^2 D_i}{\partial t^2} + \frac{\partial}{\partial t} \{ \eta(\|D\|) D_i \} \] (1.3.65)

However, in the absence of a prescribed charge density, \( \nabla \cdot D = 0 \) and, therefore,
\[ \nabla \cdot E = \nabla \cdot (\lambda(\|D\|) D) = \nabla \lambda(\|D\|) \cdot D \] (1.3.66)

The relevant nonlinear wave equation for the components \( D_i \) of the electric displacement field, i.e., (1.3.59), now follows as an immediate consequence of (1.3.65), (1.3.66), and the constitutive relation \( E_i = \lambda(\|D\|) D_i \).

Consequences of (1.3.59), as well as of the first-order systems that result by coupling Maxwell's equations to constitutive hypotheses of the form (1.3.58), will be explored, in detail, in the next chapter, as well as in Chapter 3.

### 1.4 Transmission Lines: Basic Concepts

An electrical circuit, or transmission line, is an assemblage of electrical conductors (usually in the form of wires) through which current from a power source, such as a battery or generator, flows. The components may be connected one after another (in series) or side by side (in parallel). In order to better understand the nature of a transmission line, we begin by examining, in detail, each of the constituent components of a circuit, starting with the concept of a capacitor.

A capacitor consists of two conductors that can store equal and opposite charges, say \( \pm Q \), with a potential difference between them which is independent of whether other conductors in the system are charged. In such a situation, the potential which is contributed to each of the pair of conductors by other charges must be the same. It is easily shown that the potential difference between the conductors of a capacitor is directly proportional to the charge stored, so that
\[ Q = C \Delta \phi, \] (1.4.1)

\( \Delta \phi = \phi_1 - \phi_2 \) being the potential difference between the two conductors which form the capacitor. The quantity \( C \) is called the capacitance of the capacitor and for now, we will assume it to be a constant; in our later work (i.e., in Chapters 4 and 5), we
will consider a more general situation in which \( Q = Q(V) \), \( V \) being the voltage, or potential difference, and then \( C = C(V) \) will be given by \( C = \frac{dQ}{dV} \). The simplest configuration for a capacitor is that achieved in the parallel plate capacitor, idealized to consist of two oppositely charged parallel plates in which the plate separation \( \lambda \) is small in comparison with the dimensions of each of the plates. Assuming that the region between the plates is filled by a dielectric with constant permittivity \( \epsilon \), the electric field between the plates is given by \( E = \frac{Q}{\epsilon A} \), \( A \) being the area of either of the two plates. As the potential difference between the plates is given by \( \Delta \phi = EA \), we obtain for the capacitance of this parallel plate capacitor \( C = \epsilon A/\lambda \).

When a capacitor appears as a component in an electric circuit, it is commonly indicated by the symbol \( \cap \) and capacitors in a circuit may be joined by either a series or parallel connection, each of which is depicted in Figure 1.2.

\[
\begin{align*}
\text{(a)} & \quad C_2 \quad \text{and} \quad \Delta \phi \\
\text{(b)} & \quad C_1 \quad \text{and} \quad C_2 \quad \Delta \phi
\end{align*}
\]

Fig. 1.2: (a) Series connection, (b) Parallel connection

Once two (or more) capacitors are connected, either in series or in parallel, we may speak of the combined capacitance of the combination: for two capacitors connected in series, the law of conservation of charge implies that each of the two capacitors shown in Figure 1.2(a) must acquire the same charge and this, in turn, gives an equivalent net capacitance \( C \) which satisfies \( C^{-1} = C_1^{-1} + C_2^{-1} \); on the other hand, in the case of two capacitors connected in parallel, the voltage \( \Delta \phi \), depicted in Figure 1.2(b), which acts across each of \( C_1 \) and \( C_2 \), must act across the combination so that \( C = C_1 + C_2 \).

Having considered, briefly, the concept of a capacitor, we now turn our attention to the equally important concept of a resistor. In § 1.2 we noted the experimental fact that in metals at constant temperature, the current density \( J \) is linearly proportional to the electric field \( E \), a fact which is embodied in Ohm's law \( J = \sigma E \), \( \sigma \) being the conductivity. Furthermore, for a homogeneous wire of uniform cross section, which
conducts electricity according to the linear Ohm’s law, and has its ends maintained at a constant potential difference $\Delta \phi$, we showed (i.e., (1.2.12)) that the current $I$ in the wire was given by $I = \Delta \phi / R$, where $R$, the resistance of the wire, is computed as $R = \sigma A / l$, $A$ being the cross-sectional area of the wire and $l$ its length.

We now suppose that our electric circuit or transmission line carries a steady current, i.e., we will consider, initially, direct current circuits. The basic problem of transmission line analysis is to find the current in each branch of an electric circuit given the resistance and applied voltage in each branch; the applied voltages are usually produced by a battery and although one can visualize an ideal source in the line which would provide for an applied voltage $V$, say, which is independent of the current drawn from the source, to some degree the voltage $V$ provided by the source in the circuit always exhibits a dependence $V = V(I)$.

One may define a resistor as a conducting object that is characterized by its resistance $R$ which was given, above, for a homogeneous wire of uniform cross-section, by $R = \sigma A / l$. When resistors appear in a circuit they are denoted by the symbol $\\ \ \vdash \ $ and such elements may, as was the case with capacitors, be connected either in series or parallel to form what is usually called a resistance network; these two possibilities are depicted in Figure 1.3. In the case of a series connection of two resistors the same current must pass through each resistor; applying the rule $V = IR$ to each resistor, we find that $V = IR_1 + IR_2 \equiv IR$, so that the equivalent resistance is just $R = R_1 + R_2$. In a parallel connection of the two resistors the potential drop across each resistor must be the same, while the net current through the combination is just $I = I_1 + I_2$; this implies that $I = \frac{V}{R_1} + \frac{V}{R_2} \equiv \frac{V}{R}$ so that the equivalent resistance of the combination follows as $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

Direct current electric circuits are subject to two basic rules which are known as Kirchhoff’s laws, namely

(i) The algebraic sum of all currents which flow toward a branch point is zero, where a branch point is a point of the circuit joining three or more conductors, and

(ii) The algebraic sum of the voltage differences around any loop is zero, where a loop refers to any closed conducting path in the circuit; this second of Kirch-
Kirchhoff's laws is equivalent to the statement that the sum of the current-resistance products $\sum_{j} I_{j} R_{j}$ around any closed path must equal the total electromotive force in the loop.

Fig. 1.3: (a) Series connection of two resistors; (b) Parallel connection of two resistors

In short order we will generalize the above-referenced Kirchhoff's laws so as to take into account slowly varying currents.

We now turn to the important circuit parameter known as the self-inductance, which arises from the relationship between the flux and current associated with an isolated circuit. The concept of magnetic flux was introduced in § 1.2 (i.e., (1.2.44)) as $\Phi = \int_{S} \mathbf{B} \cdot \mathbf{n} \, da$. By virtue of the Biot-Savart law (1.2.30), the magnetic flux linking an isolated circuit is linearly dependent on the current in the circuit and, therefore, in a rigid stationary circuit, the only changes in $\Phi$ which are possible are those which arise from changes in the current; we may write that

$$\frac{d\Phi}{dt} = \frac{d\Phi}{dI} \frac{dI}{dt} \equiv L \frac{dI}{dt} \quad (1.4.2)$$
where \( L \equiv \frac{d\Phi}{dt} \), which is called the inductance, will be assumed, for our purposes in this section, to be a constant. In terms of the inductance \( L \) we may rewrite Faraday's law (1.2.56) in the form

\[
\mathcal{E} = -L \frac{dI}{dt}
\]  

(1.4.3)

The mks unit of inductance is the henry (denoted by \( \text{H} \)) which is equal, as a consequence of (1.4.3), to one volt-second/ampere, since the unit of emf \( \mathcal{E} \) in the mks system is the volt. The reader may easily locate in standard texts on electromagnetic theory [134], [73] the calculation of self-inductance for several important simple circuits, e.g., the self-inductance of a toroidal coil of constant cross-sectional area \( A \), wound with \( N \) turns of wire of mean length \( l \), and carrying a current \( I \) is given by

\[
L = \frac{\mu_0 N^2 A}{l}
\]  

(1.4.4)

where (1.4.4) follows from (1.4.2) and the fact that the total flux linking the \( N \) turns is

\[
\Phi = \frac{\mu_0 N^2 A}{l} I
\]  

(1.4.5)

Inasmuch as we will only consider isolated circuits in our work in Chapters 4 and 5, we shall not indulge here in a discussion of the interesting topic of mutual inductance, which must be considered when several circuits are involved and the emf induced in the \( i \)-th circuit results from current changes in all the circuits present. As is the case with resistors and capacitors, inductances are often connected in series (and in parallel) in a circuit and it is possible to compute an effective inductance in each case. In such situations, however, one must take into account the coupling which occurs between the inductors, i.e., for two inductors of strengths \( L_1 \) and \( L_2 \) in series, there is a mutual inductance of strength

\[
M = \kappa \sqrt{L_1 L_2}, \quad -1 \leq \kappa \leq 1
\]  

(1.4.6)

which must be taken into account and which leads to an effective inductance for such a series circuit of the form

\[
L_{\text{eff}} = L_1 + L_2 + 2\kappa \sqrt{L_1 L_2}
\]  

(1.4.7)
For two inductors in parallel, again with strengths $L_1$ and $L_2$, the result is even somewhat more complicated than (1.4.7), namely,

$$L_{\text{eff}} = \frac{(L_1L_2 - M^2)}{(L_1 + L_2 - 2M)} \quad \text{(1.4.8)}$$

As we shall find sufficient difficulties with the consequences of nonlinearity in the discussion of distributed parameter transmission lines in Chapters 4 and 5, we will bypass the derivation of the effective self-inductances (1.4.7) and (1.4.8) and confine our attention henceforth to circuits involving a single inductor.

To this point we have only considered circuits involving currents which are excited by a constant applied voltage. If instead of a constant applied voltage we have to deal with a slowly varying voltage, then a slowly varying current will arise in response, provided the line does not radiate away a considerable amount of electromagnetic energy. When such slowly varying applied voltages change periodically with time, it is found that an appreciable time after the application of such voltages, the currents in the line also vary periodically with time and the discussion of the behavior of such circuits is usually dependent on whether it is the periodic or the nonperiodic behavior that is of prime importance. The intrinsic behavior of an electrical circuit with a nonperiodic evolution of current is referred to as the steady-state behavior; the simplest possible instances of each type of behavior involve the transient analysis associated with excitation by a constant applied voltage and the steady-state analysis associated with excitation by a periodically (e.g., sinusoidal) varying applied voltage.

For slowly varying currents not only resistors, but also capacitors and inductors, must be included as circuit elements and each such element in the line also involves a potential difference which must be included in Kirchhoff's loop law; the terminology "counter voltage" has been applied most often in the literature to describe the potential difference between the terminals of passive elements such as these. Noting that each of Kirchhoff's laws has to apply to the instantaneous values of the currents, counter voltages, and applied voltages in a circuit, we may restate these laws, for slowly varying currents, as follows:

(i) The algebraic sum of the instantaneous currents which flow toward a branch point is zero.
(ii) The algebraic sum of the instantaneous applied voltages in a closed loop is equal to the algebraic sum of the instantaneous counter voltages in the loop.

As a simple example of the appropriate interpretation of the second of Kirchhoff's laws, above, for the case of slowly varying current in a circuit, we may consider the simple closed loop indicated in Figure 1.4 below:

The basic sign convention for the circuit depicted in Figure 1.4 is that the applied voltage $V(t)$, which is shown as being in series with a resistance $R$, an inductance $L$, and a capacitance $C$, is positive if it tends to induce the current to move in the indicated direction, i.e., clockwise in this case. The counter voltage due to the resistor is, of course, just $IR$, while that due to the inductor is $L \frac{dI}{dt}$; the capacitance counter voltage is the ratio $\frac{Q}{C}$, where $Q = \int_{t_0}^{t} I(\tau) d\tau$. Therefore, the second of Kirchhoff's laws for the elementary circuit shown in Figure 1.4 reads

$$V(t) = IR + L \frac{dI}{dt} + \frac{1}{C} \int_{t_0}^{t} I(\tau) d\tau$$

(1.4.9)

Now, consider the circuit of Figure 1.4 modified so as to include a battery which is capable of producing a constant applied voltage $V_0$ and a switch which we denote, in Figure 1.5 below, by $S$:
For this series RLC circuit, the relation (1.4.9) applies, so that after the switch $S$ is closed we have

$$V_0 = IR + L \frac{dI}{dt} + \frac{1}{C} \int_0^t I(\tau) d\tau$$

(1.4.10)

where, as $t_0$ may be interpreted as that time at which the charge on the capacitor is zero, we have simply taken $t_0 = 0$. Differentiation of (1.4.10) once with respect to time yields the familiar second-order linear ODE

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = 0$$

(1.4.11)

which has the solution

$$I(t) = (\alpha e^{i\omega_n t} + \beta e^{-i\omega_n t}) e^{-\frac{R}{2L} t}$$

(1.4.12)

with

$$\omega_n = \left( \frac{1}{LC} - \frac{R^2}{4L^2} \right)^{1/2}$$

(1.4.13)

In view of the fact that the current $I$ should be real-valued, the constant $\beta$ in (1.4.12) must be the complex conjugate of $\alpha$; also $I(0) = 0$ because the switch $S$ is closed at $t = 0$. These considerations reduce (1.4.12) to

$$I(t) = \gamma \sin \omega_n t \left( e^{-\frac{R}{2L} t} \right)$$

(1.4.14)
Evaluating (1.4.10) at $t = 0$, we readily find that $V_0 = \frac{L}{dt} |_{t=0}$, in which case
\[ \gamma = \frac{V_0}{\omega_n L} = \frac{V_0}{\left(\frac{B}{L} - \frac{R^2}{4}\right)^{1/2}} \quad (1.4.15) \]

Relations (1.4.14), (1.4.15) completely characterize the transient response of the basic series RLC circuit depicted in Figure 1.5; the current in this circuit possesses an amplitude $\gamma e^{-\frac{B}{2}t}$, which is exponentially decreasing in time, and the current oscillates with the natural frequency given by (1.4.13).

Having characterized the transient response of an RLC circuit we now return to the situation depicted in Figure 1.4 and consider the steady-state behavior of such a circuit subject to an excitation of the form
\[ V(t) = V_0 \cos \omega t \quad (1.4.16) \]
where $\omega$ is a given frequency. If we note that $V(t) = \text{Re} V_0 e^{i\omega t}$, then we may consider a complex voltage of the form $V_1(t) + iV_2(t)$ applied to the circuit of Figure 1.4 which results in a complex valued current $I_1(t) + iI_2(t)$ that satisfies, by virtue of (1.4.9),
\[ \frac{dV_1}{dt} + i \frac{dV_2}{dt} = \left( L \frac{d^2I_1}{dt^2} + R \frac{dI_1}{dt} + \frac{1}{C} I_1 \right) + i \left( L \frac{d^2I_2}{dt^2} + R \frac{dI_2}{dt} + \frac{1}{C} I_2 \right) \quad (1.4.17) \]
Thus each of $I_1(t), I_2(t)$ satisfies
\[ \frac{dV}{dt} = L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I \quad (1.4.18) \]
with, respectively, $V_1(t), V_2(t)$ appearing on the left-hand side of this equation. It is, therefore, sufficient in the present situation to solve (1.4.18), with $V(t) = V_0 e^{i\omega t}$, for $I(t) = I_0 e^{i\omega t}$, $I_0$ a complex constant; the physical current will then be $\text{Re} I_0 e^{i\omega t}$. Substitution of $V(t) = V_0 e^{i\omega t}$, $I(t) = I_0 e^{i\omega t}$ into (1.4.18) produces the relation
\[ i\omega V_0 e^{i\omega t} = \left[ -\omega^2 L + i\omega R + \frac{1}{C} \right] I_0 e^{i\omega t} \quad (1.4.19) \]
or
\[ V_0 e^{i\omega t} = Z I_0 e^{i\omega t} \quad (1.4.20) \]
where the quantity $Z$, the impedance of the circuit, is given by

$$Z = R + i \left( \omega L - \frac{1}{\omega C} \right) \quad (1.4.21)$$

We note that $Z$ consists of a real part, the resistance $R$, and an imaginary part $X = \omega L - \frac{1}{\omega C}$ which is usually referred to as the reactance. Because $Z$ is not, in general, real-valued, it follows, as a consequence, that the steady-state current in the RLC circuit will not be in phase with the applied voltage $V(t)$; in fact, if we define

$$\left\{ \begin{array}{l}
|Z| = \left( R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2 \right)^{1/2} \\
\theta = \tan^{-1} \left( \frac{\omega L - \frac{1}{\omega C}}{R} \right)
\end{array} \right. \quad (1.4.22)$$

then $Z = |Z|e^{i\theta}$ and substitution of this form of the impedance into (1.4.20) produces

$$I(t) = \frac{V_0}{|Z|} e^{i(\omega t - \theta)} \quad (1.4.23)$$

so that the physical current in the circuit, which is now obtained by taking the real part of both sides of (1.4.23), is given by

$$I_p(t) = \frac{V_0}{|Z|} \cos(\omega t - \theta) \quad (1.4.24)$$

For $\theta > 0$ the current $I_p(t)$, in the circuit, will achieve a prescribed phase at a later time than the voltage, in which case we say that the current lags the voltage; the opposite situation applies, of course, if $\theta < 0$.

This concludes our discussion of the elementary transient and steady-state behavior of the simple RLC series circuit in which the inductance $L$ and the capacitance $C$ have been assumed to be constants. In our work in Chapters 4 and 5 on the behavior of nonlinear transmission lines, we will not only allow for the possibility that the charge in the line (and hence the capacitance) is voltage dependent, but will also introduce into the circuits under consideration a nonlinear, voltage dependent leakage conductance.
Bibliography


BIBLIOGRAPHY


